An Invariant Measure for the Loop Space of a Simply Connected Compact Symmetric Space

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Abstract. Let X denote a simply connected compact Riemannian symmetric space, U the universal covering of the identity component of the group of automorphisms of X, and LU the loop group of U. In this paper we prove the existence (and conjecture the uniqueness) of an LU-invariant probability measure on a distributional completion of the loop space of X.

§0. Introduction.

Let K denote a simply connected compact Lie group, and let G denote the complexification of K. In [Pi1] we proved the following

Theorem (0.1). *There exists a $L_{pol}K$-biinvariant probability measure μ on the formal completion, $L_{formal}G$, of the complex loop group LG.*

We also conjectured that there is a unique such biinvariant measure.

One purpose of this paper is to present a more transparent proof of this theorem, especially in the simplest case, $K = SU(2)$ (see §3). A second purpose is to generalize the theorem to a context in which K is replaced by a simply connected compact symmetric space X. Before stating this generalization, we will clarify the meaning of various terms.

First $L_{pol}K$ and $L_{an}K$ denote the groups consisting of maps from S^1 into K which are polynomial (i.e. have finite Fourier series) and real analytic, respectively, with pointwise multiplication. The complexified loop group, $L_{an}G = H^0(S^1, G)$, is a complex Lie group. A neighborhood of the identity consists of those loops which have a unique (triangular or Birkhoff or Riemann-Hilbert) factorization

$$g = g_- \cdot g_0 \cdot g_+, \tag{0.2}$$

where $g_- \in H^0(D^*, \infty; G, 1)$, $g_0 \in G$, $g_+ \in H^0(D, 0; G, 1)$, and D and D^* denote the *closed* disks centered at 0 and ∞, respectively (thus (0.2) is an equality of holomorphic functions which holds on a thin collar of S^1, the collar depending upon g). A model for this neighborhood is

$$H^1(D^*, \mathfrak{g}) \times G \times H^1(D, \mathfrak{g}), \tag{0.3}$$

where the linear coordinates are determined by $\theta_+ = g_+^{-1}(\partial g_+)$, $\theta_- = (\partial g_-)g_-^{-1}$. The (left or right) translates of this neighborhood by elements of $L_{pol}K$ cover $H^0(S^1, G)$.

The hyperfunction completion, $L_{hyp}G$, is modelled on the space

$$H^1(\Delta^*, \mathfrak{g}) \times G \times H^1(\Delta, \mathfrak{g}), \tag{0.4}$$

where Δ and Δ^* denote the *open* disks centered at 0 and ∞, respectively, and the transition functions are obtained by continuously extending the transition functions for the analytic loop space of the preceding paragraph. The global definition is

$$L_{hyp}G = \lim_{r \downarrow 1} H^0(\{1 < |z| < r\}, G) \times_{H^0(S^1,G)} \lim_{r \uparrow 1} H^0(\{r < |z| < 1\}) \tag{0.5}$$

From the global definition it is clear that the group $H^0(S^1, G)$ acts naturally from both the left and right of $L_{hyp}G$.

The formal completion is defined in a similar way, where $H^1(\Delta, \mathfrak{g})$ is replaced by the corresponding space of formal power series

$$H^1_{formal}(\Delta, \mathfrak{g}) = \{\theta_+ = (\theta_1 + \theta_2 z + ..)dz, \quad \theta_i \in \mathfrak{g}\} \simeq \prod_1^\infty \mathfrak{g}. \tag{0.6}$$

The global definition of the formal completion is

$$\mathbf{L}G = L_{formal}G = G(\mathbb{C}((z^{-1}))) \times_{G(\mathbb{C}(z))} G(\mathbb{C}((z))), \tag{0.7}$$

where $\mathbb{C}((z))$ is the field of formal Laurent series $\sum a_n z^n$, $a_n = 0$ for $n << 0$, and $G(\mathbb{C}(z)) = L_{pol}G$.

To summarize, there is an equivariant way to form various "distributional" completions of the complex loop group, respecting its homotopy and complex structure,

$$L_{pol}G \to L_{an}G \to L_{C^0 \cap W^{1/2}}G \to L_{hyp}G \to \mathbf{L}G \tag{0.8}$$

where $\mathbf{L}G$ and $L_{hyp}G$ are "dual" to $L_{pol}G$ and $L_{an}G$, respectively, and $L_{C^0\cap W^{1/2}}G$ is essentially "self-dual".

An aesthetic advantage of the hyperfunction completion is that $\mathcal{D}$, the group of analytic orientation-preserving diffeomorphisms of S^1, acts equivariantly. If μ is supported on $L_{hyp}G$, then the conjectured uniqueness of μ implies the conjecture that μ is $\mathcal{D}$-invariant.

Now suppose that X is a simply connected compact symmetric space with a fixed basepoint. From this we obtain groups as in the following diagram,

$$\begin{array}{ccccc} & & G & & \\ & \nearrow & & \nwarrow & \\ G_{\mathbb{R}} & & & & U\,, \\ & \nwarrow & & \nearrow & \\ & & K & & \end{array} \tag{0.10}$$

where U is a simply connected compact group acting isometrically and transitively on X, $X \simeq U/K$, G is the complexification of U, and $G_{\mathbb{R}}$ acts by automorphisms of the dual (nonunitary type) symmetric space $G_{\mathbb{R}}/K$. There are equivariant totally geodesic embeddings

$$\begin{array}{ccc} U/K & \to & U \\ \downarrow & & \downarrow \\ G/G_{\mathbb{R}} & \to & G \end{array} \tag{0.11}$$

of symmetric spaces. The main point is that the antiholomorphic involution which fixes $L(G/G_{\mathbb{R}}) \to LG$ extends naturally to our distributional completions of LG. We define the hyperfunction and formal completions of $L(G/G_{\mathbb{R}})$ as the identity components of the fixed point sets of these extensions. We obtain equivariant homotopy equivalences

$$\begin{aligned} L_{pol}(U/K) \to L_{pol}(G/G_{\mathbb{R}}) \to L_{an}(G/G_{\mathbb{R}}) \\ \to L_{hyp}(G/G_{\mathbb{R}}) \to \mathbf{L}(G/G_{\mathbb{R}}). \end{aligned} \tag{0.12}$$

The main result is the following

Theorem (0.13). *There exists a $L_{pol}U$-invariant probability measure on the formal completion, $\mathbf{L}(G/G_{\mathbb{R}})$.*

We conjecture that there is a unique such invariant measure.

A general simply connected X can be written uniquely as a product of irreducible X (irreducible means that $\mathfrak{u}$ does not have any Θ-invariant ideals, where Θ is the involution of U which fixes K). There are two types of irreducible X.

In the type I case $\mathfrak{u}$ is simple. In the type II case $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{k}$, where $\mathfrak{k}$ is simple, $\Theta(x, y) = (y, x)$, and $X = K$. Thus the essential novelty of this paper concerns the type I case.

The plan of the paper is the following. In §1 and §2 we introduce the basic notation used throughout the paper. In §1 we review some basic facts about triangular decompositions and symmetric spaces (readers interested in basic cases such as S^2 and $SU(2)$ can probably skip this section entirely). A more detailed treatment of the statements in this section will appear in [Pi4]. In §2 we consider loop spaces and their completions, and prove the minimal structural results that we will need.

In §3 we will give a relatively elementary proof of Theorem (0.1) in the case $K = SU(2)$. This proof reveals that we can a priori compute many distributions for the measure μ in Riemann-Hilbert coordinates, (0.4). We briefly discuss the issue of uniqueness, but this remains unresolved.

In §4 we will give a corresponding elementary proof of Theorem (0.13) in the case S^2.

In §5 and §6 we prove Theorem (0.13) in general. The proofs in §3-§6 depend upon a certain compactness result for Wiener measures parameterized by temperature (or radius of the circle); this depends heavily on ideas of the Malliavins ([MM]).

In §7 we discuss a conjectural formula for the "diagonal distribution", and some of its potential implications.

§1. Symmetric Spaces and Triangular Decompositions.

Throughout the remainder of this paper, U will denote a simply connected compact Lie group, Θ will denote an involution of U, with fixed point set K, and X will denote the quotient, U/K. The space X has the structure of a simply connected symmetric space of compact type. Conversely, given such a space X, together with the choice of basepoint, there is a symmetric pair (U, K), satisfying the conditions above, such that $X \simeq U/K$. To pin down U in terms of X, we could choose U to be the universal covering of the identity component of the group of automorphisms of X; but for technical reasons, we will not assume this at the outset.

There is a unique extension of Θ to a holomorphic automorphism of G, the complexification of U. Corresponding to the diagram of groups in (0.10), there is a

Lie algebra diagram

$$\begin{array}{ccccc} & & \mathfrak{g}=\mathfrak{u}\oplus i\mathfrak{u} & & \\ & \nearrow & & \nwarrow & \\ \mathfrak{g}_{\mathbb{R}}=\mathfrak{k}\oplus\mathfrak{p} & & & & \mathfrak{u}=\mathfrak{k}\oplus i\mathfrak{p} \\ & \nwarrow & & \nearrow & \\ & & \mathfrak{k} & & \end{array} \quad , \tag{1.1}$$

where Θ, acting on the Lie algebra level, is $+1$ on $\mathfrak{k}$ and -1 on $\mathfrak{p}$. We let $(\cdot)^{-*}$ denote the Cartan involution for the pair (G,U) (so that $(\cdot)^*$ is an antiholomorphic antiinvolution). The Cartan involution for the pair $(G,G_{\mathbb{R}})$ is given by $\tau(g)=g^{-*\Theta}$. Since $*$, Θ, τ, and $(\cdot)^{-1}$ commute, our practice of writing g^{Θ} for $\Theta(g)$, etc, should not cause any confusion.

There are natural maps

$$\begin{array}{ccccc} K & \to & U & \to & U/K \\ \downarrow & & \downarrow & & \downarrow \\ G_{\mathbb{R}} & \to & G & \to & G/G_{\mathbb{R}} \end{array} . \tag{1.2}$$

The vertical arrows (given by inclusion) are homotopy equivalences; more precisely, there are diffeomorphisms (polar or Cartan decompositions)

$$K\times\mathfrak{p}\to G_{\mathbb{R}}, \quad U\times i\mathfrak{u}\to G, \quad U\times_K i\mathfrak{k}\to G/G_{\mathbb{R}}, \tag{1.3}$$

in each case given by the formula $(g,X)\to gexp(X)$ (mod $G_{\mathbb{R}}$ in the last case). In turn there are totally geodesic embeddings (or morphisms) of symmetric spaces

$$\begin{array}{ccccccc} U/K & \xrightarrow{\phi} & U & : & gK & \to & gg^{-\Theta} \\ \downarrow & & \downarrow & & & & \\ G/G_{\mathbb{R}} & \xrightarrow{\phi} & G & : & gG_{\mathbb{R}} & \to & gg^{*\Theta} \end{array} \tag{1.4}$$

The map ϕ is equivariant, where $g\in G$ acts on $g_0\in G$ by $g_0\to gg_0g^{*\Theta}$. The ϕ-images are defined by simple algebraic equations, modulo connectedness issues,

$$\begin{array}{ccc} \phi(U/K)=\{g\in G: g^{-1}=g^*=g^{\Theta}\}_0 & \to & U=\{g^{-1}=g^*\} \\ \downarrow & & \downarrow \\ \phi(G/G_{\mathbb{R}})=\{g^*=g^{\Theta}\}_0 & \to & G \end{array} \tag{1.5}$$

where the 0-subscript denotes the connected component containing the identity (Note: the maps (1.4) exist for arbitrary automorphisms Θ; the simple characterization of the images is peculiar to automorphisms of order 2).

Fix a maximal abelian subalgebra $\mathfrak{t}_0 \subset \mathfrak{k}$. We then obtain Θ-stable Cartan subalgebras

$$\mathfrak{h}_0 = \mathcal{Z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{t}_0) = \mathfrak{t}_0 \oplus \mathfrak{a}_0, \quad \mathfrak{t} = \mathfrak{t}_0 \oplus i\mathfrak{a}_0 \tag{1.6}$$

and $\mathfrak{h} = \mathfrak{h}_0^{\mathbb{C}}$, for $\mathfrak{g}_{\mathbb{R}}$, $\mathfrak{u}$, and $\mathfrak{g}$, respectively, where $\mathfrak{a}_0 \subset \mathfrak{p}$. We let T_0 and T denote the maximal tori in K and U corresponding to $\mathfrak{t}_0$ and $\mathfrak{t}$, respectively.

Let Δ denote the roots for $\mathfrak{h}$ acting on $\mathfrak{g}$; $\Delta \subset \mathfrak{h}_{\mathbb{R}}^*$, where $\mathfrak{h}_{\mathbb{R}} = \mathfrak{a}_0 \oplus i\mathfrak{t}_0$. We choose a Weyl chamber C^+ which is Θ-stable (to prove that C^+ exists, we must show that $i\mathfrak{t}_0$, the $+1$ eigenspace of Θ acting on $\mathfrak{h}_{\mathbb{R}}$, intersects the regular part of $\mathfrak{h}_{\mathbb{R}}$; since $\mathfrak{t}_0$ is maximal abelian, we can find regular elements in $\mathfrak{t}_0$). Since $\tau = -(\cdot)^{*\Theta}$ and $(\cdot)^*$ is the identity on $\mathfrak{h}_{\mathbb{R}}$, $\tau(C^+) = -C^+$.

Given our choice of C^+, we obtain a Θ-stable triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, so that $\tau(\mathfrak{n}^{\pm}) = \mathfrak{n}^{\mp}$. Let $N^{\pm} = exp(\mathfrak{n}^{\pm})$, $H = exp(\mathfrak{h})$, and $B^{\pm} = HN^{\pm}$. We also let W denote the Weyl group, $W = N_U(T)/T \simeq N_G(H)/H$.

At the group level we have the Birkhoff or triangular or LDU decomposition for G,

$$G = \bigsqcup_W \tilde{\Sigma}_w, \quad \tilde{\Sigma}_w = N^- wHN^+, \tag{1.7}$$

where $\tilde{\Sigma}_w$ is diffeomorphic to $(N^- \cap wN^-w^{-1}) \times H \times N^+$. The intersection of this decomposition with the spaces in (1.5) is considered in detail in [Pi4]. Here we will only prove what we need for $\phi(G/G_{\mathbb{R}})$.

(1.8) Proposition. *Fix $w \in W$.*

(a) The intersection $\{g^* = g^\Theta\} \cap \tilde{\Sigma}_w$ is nonempty if and only if there exists $\mathbf{w} \in w \subset N_U(T)$ such that $\mathbf{w}^{*\Theta} = \mathbf{w}$; $\mathbf{w}$ is unique modulo the action

$$T \times \{\mathbf{w} \in N_U(T) : \mathbf{w}^{*\Theta} = \mathbf{w}\} \to \{\mathbf{w}^{*\Theta} = \mathbf{w}\} : \lambda, \mathbf{w} \to \lambda \mathbf{w} \lambda^{*\Theta}.$$

(b) The orbits of B^- in $\{g^* = g^\Theta\} \cap \tilde{\Sigma}_w$ are open and indexed by

$$\pi_0(\{\mathbf{w} \in w : \mathbf{w}^{*\Theta} = \mathbf{w}\}) \simeq \{\mathbf{w} \in w : \mathbf{w}^{*\Theta} = \mathbf{w}\}/T.$$

(c) The B^--orbit through $\mathbf{w}$ belongs to $\phi(G/G_{\mathbb{R}})$ if and only if $Ad(\mathbf{w}) \circ \Theta$ is equivalent to Θ through conjugation by $Ad(K)$.

(d) For the top stratum, the map

$$N^- \times (T_0^{(2)} \times_{T_0^{(2)} \cap exp(i\mathfrak{a}_0)} exp(i\mathfrak{a}_0)) \times exp(i\mathfrak{t}_0) \to \{g^* = g^\Theta\} \cap \tilde{\Sigma}_1$$

$$l, [\mathbf{w}, m], a_\phi \to g = l\mathbf{w}ma_\phi l^{*\Theta}$$

is a diffeomorphism.

In this paper we will only use (d). To prove this suppose that $g \in \tilde{\Sigma}_1$. There is a unique decomposition $g = lhu$, where $l \in N^-$, $h \in H$, and $u \in N^+$. If $g = g^{*\Theta}$, then $lhu = u^{*\Theta}h^{*\Theta}l^{*\Theta}$, and because $(\cdot)^{*\Theta}$ interchanges $N^{\pm}$, uniqueness of the decomposition implies $u = l^{*\Theta}$ and $h = h^{*\Theta}$. This leads to (d) in a routine way.

Proposition (1.10). *For the action of U on G given by $u, g \to ugu^{-\Theta}$, for each U-orbit $\mathcal{O}$, the intersection of $\mathcal{O}$ with $\tilde{\Sigma}_1$ is dense in $\mathcal{O}$.*

Proof of (1.10). Let v denote a highest weight vector for the highest weight representation corresponding to the sum of the dominant integral functionals. Then

$$\tilde{\Sigma}_1 = \{g \in G : \langle g \cdot v, v\rangle \neq 0\}, \tag{1.11}$$

where $\langle \cdot, \cdot \rangle$ denotes the essentially unique U-invariant Hermitian inner product.

Fix $g_0 \in G$. We must show that there exist $g \in U$, arbitrarily close to 1, such that $\langle g g_0 g^{-\Theta} v, v\rangle \neq 0$. Suppose that this is not the case. Then

$$\langle g g_0 g^{-\Theta} v, v\rangle = 0, \tag{1.12}$$

for all $g \in U$ sufficiently close to 1. Since the left hand side of this equation is a holomorphic function of $g \in G$, and since U is a real form of G, (1.12) holds for all $g \in G$. If we take $g = b \in B^+$, then $b^{-\Theta} \in B^+$, $b^{-\Theta} \cdot v$ is a multiple of v, and $b^* \in B^-$. Hence (1.12) implies

$$\langle g_0 v, b^* v\rangle = 0, \quad \forall b^* \in B^-. \tag{1.13}$$

But $\{b^* v : b^* \in B^-\}$ spans the space of the representation. This is a contradiction. $\square$

Corollary (1.14). *Let U act on G as in* (1.10). *Suppose that ν is a U-quasi-invariant measure on G. Then the ν-measure of the complement of $\tilde{\Sigma}_1$ is zero.*

Proof of (1.14). The measure ν will have a disintegration along the orbits of U, and for *a.e.* such orbit, the derivative $\nu_{\mathcal{O}}$ will belong to the unique (Lebesgue) invariant measure class. Proposition (1.10) implies that the $\nu_{\mathcal{O}}$-measure of $\mathcal{O} \setminus \tilde{\Sigma}_1$ is zero. This implies (1.14). $\square$

§2. Loop Spaces and Completions.

In the introduction we recalled the definitions of $\mathbf{L}G$ and $L_{hyp}G$ (see Part III, chapter 2, of [Pi1]). To consider the finer properties of loop groups and their completions, it is useful to adopt the Kac-Moody point of view, where many structural results carry over from the theory of simple Lie algebras. This point of view will be needed only occasionally (in the proof of Lemma (6.5) and Appendix B). Otherwise it will suffice to adopt a more naive approach, developed here.

These completions have generalized Birkhoff factorizations

$$\mathbf{L}G = \bigsqcup_{Hom(S^1,T)} \tilde{\Sigma}_\lambda^{formal}, \quad \tilde{\Sigma}_\lambda^{formal} = G(\mathbb{C}[[z^{-1}]]) \cdot \lambda \cdot G(\mathbb{C}[[z]]) \tag{2.1}$$

where $\mathbb{C}[[\zeta]]$ denotes formal power series in ζ, and

$$L_{hyp}G = \bigsqcup_{Hom(S^1,T)} \tilde{\Sigma}_\lambda^{hyp}, \quad \tilde{\Sigma}_\lambda^{hyp} = H^0(\Delta^*, G) \cdot \lambda \cdot H^0(\Delta, G), \tag{2.2}$$

which restrict to the standard Birkhoff factorization of LG (see chapter 8 of [PS]). It is the existence of these coherent decompositions, corresponding to different smoothness conditions, which imply that these various completions are all homotopy equivalent (see (8.6.6) in [PS]).

In both the formal and hyperfunction cases, the top stratum (the piece with $\lambda = 1$ above) is open and dense, and for each point g in the top stratum, there is a unique factorization as in (0.2), where in the hyperfunction case $g_\pm$ are G-valued holomorphic functions in the the open disks Δ and Δ^*, respectively, and in the formal case $g_\pm$ are simply formal power series satisfying the appropriate algebraic equations determined by G. We will refer to g_-, g_0, g_+ (θ_-, g_0, θ_+, respectively) as the Riemann-Hilbert coordinates (linear Riemann-Hilbert coordinates, respectively) of g.

Both $(\cdot)^*$ and Θ act on LG, by acting on a loop pointwise.

Lemma (2.3). *Both $(\cdot)^*$ and Θ extend continuously to involutions of $L_{hyp}G$ and $\mathbf{L}G$, and in general both permute the Birkhoff strata.*

Proof of (2.3). We will consider the hyperfunction case for definiteness. Suppose that $g \in L_{hyp}G$. This means that $g = [g_l, g_r]$ is an equivalence class represented by a pair (g_l, g_r), where g_l (g_r) is a holomorphic map $\{1 - \epsilon < |z| < 1\} \to G$ ($\{1 < |z| < 1 + \epsilon\} \to G$, respectively), for some $\epsilon > 0$, and this pair is equivalent to any other pair of the form $(g_l h, h^{-1} g_r)$, for some $h \in H^0(S^1, G)$. The extension of $(\cdot)^*$ is given by

$$g^* = [g_r(\bar{z}^{-1})^*, g_l(\bar{z}^{-1})^*]. \tag{2.4}$$

This is well-defined and restricts to the pointwise action of $(\cdot)^*$ on $H^0(S^1, G)$, simply because $z^{-1} = \bar{z}$ on S^1.

From the formula (2.4), it is clear that $(\cdot)^*$ maps the λ stratum to the $\lambda^* = \lambda^{-1}$ stratum.

For Θ, the extension is given by $g^\Theta = [g_l^\Theta, g_r^\Theta]$ (i.e. we are simply applying Θ pointwise). Again it is clear this is well-defined, extends the pointwise application of Θ to $H^0(S^1, G)$, and the λ stratum is mapped into the λ^Θ stratum. $\square$

Definition (2.5). *$\mathbf{L}(G/G_{\mathbb{R}})$ and $L_{hyp}(G/G_{\mathbb{R}})$ are the identity components of the fixed point sets of $(\cdot)^{*\Theta}$ acting on $\mathbf{L}G$ and $L_{hyp}G$, respectively. The intersections of $\tilde{\Sigma}_1^{formal}$ and $\tilde{\Sigma}_1^{hyp}$ with $\mathbf{L}(G/G_{\mathbb{R}})$ and $L_{hyp}(G/G_{\mathbb{R}})$, respectively, will be referred to as the top strata.*

Proposition (2.6). *(a) The intersection $\tilde{\Sigma}_\lambda^{formal} \cap \mathbf{L}(G/G_{\mathbb{R}})$ is nonempty if and only if $\lambda^{*\Theta} = \lambda$ in $Hom(S^1, T)$.*

(b) The top stratum is open and dense and diffeomorphic to

$$G/G_{\mathbb{R}} \times H^1_{formal}(\Delta, \mathfrak{g}).$$

The analogous statements hold in the hyperfunction case.

We only need Part (b). This follows directly from the uniqueness of the Birkhoff factorization when $\lambda = 1$.

Finally we recall some facts about automorphisms.

Suppose that the group U is the universal covering of the identity component of $Aut(X)$. Let π denote the projection, so that we have an exact sequence

$$0 \to ker(\pi) \to U \xrightarrow{\pi} Aut(X) \xrightarrow{\pi_0} \pi_0(Aut(X)) \to 0. \tag{2.7}$$

Let A_0 denote the identity component of $Aut(X)$. Define

$$\mathcal{L}A_0 = \{g : \mathbb{R} \to U : \exists \Delta \in ker(\pi) \quad with \quad g(t+1) = g(t)\Delta, \forall t\}, \tag{2.8}$$

where some degree of smoothness ($> 1/2$)) is implicitly fixed. There is an exact sequence

$$0 \to ker(\pi) \to \mathcal{L}A_0 \to LAut(X) \to \pi_0(Aut(X)) \to 0, \tag{2.9}$$

where $g \in \mathcal{L}A_0$ maps to the loop $\pi(g(e^{2\pi it}))$. We will indicate the degree of smoothness by attaching a subscript, e.g. $\mathcal{L}_{pol}A_0$ denotes the group of elements g such that $\pi(g(e^{2\pi it}))$ is polynomial, i.e. has a finite Fourier series relative to some matrix representation of $Aut(X)$.

It is straightforward to check that $\mathcal{L}_{pol}A_0$ acts naturally on $\mathbf{L}(G/G_{\mathbb{R}})$ (these actions will be written out explicitly in subsequent sections).

These generalized loops will play a role in this paper for the following reason. When one fixes a triangular decomposition for $\mathfrak{g}$, there is no residual $Ad(G)$ symmetry. However outer automorphisms occasionally exist that respect this decomposition. For the loop algebra, $L\mathfrak{g}$, elements of the center $C(K)$ give rise to outer automorphisms (see (3.4.4) of [PS]) that interact in a relatively simple way with Riemann-Hilbert factorization. These outer automorphisms are represented by multivalued loops.

§3. The Group Case, $X = K = SU(2)$.

In this section we will give a relatively elementary proof of Theorem (0.1), in the case $K = SU(2)$. We will then discuss a possible method for computing the measure explicitly.

Existence.

Suppose that

$$g = g_- \cdot g_0 \cdot g_+ \tag{3.1}$$

is a point in the top stratum of $\mathbf{L}SL(2,\mathbb{C})$. We write

$$g_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}, \qquad a_0d_0 - b_0c_0 = 1, \tag{3.2}$$

$$g_+ = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} = 1 + \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix} z + .. \tag{3.3}$$

and

$$g_- = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} = 1 + \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} z^{-1} + .. \tag{3.4}$$

The action of the constants $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ on $\mathbf{L}SL(2,\mathbb{C})$ is completely transparent in these coordinates: for $g_l, g_r \in SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$,

$$g_l \cdot g \cdot g_r^{-1} = [g_l g_- g_l^{-1}] \cdot [g_l g_0 g_r^{-1}] \cdot [g_r g_+ g_r^{-1}]. \tag{3.5}$$

Let σ denote the outer automorphism of $LSL(2,\mathbb{C})$ given by

$$\sigma(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a & bz \\ cz^{-1} & d \end{pmatrix}, \tag{3.6}$$

i.e. σ is conjugation by the multivalued loop $\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}$. We will write out the action of σ on $\mathbf{L}SL(2,\mathbb{C})$ in the Lemma below.

Let $\mathbf{w}_0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and

$$i_0(SL(2,\mathbb{C})) = \mathbf{w}_0 SL(2,\mathbb{C})^\sigma \mathbf{w}_0^{-1} = \{\begin{pmatrix} d & cz^{-1} \\ bz & a \end{pmatrix} : ad - bc = 1\}. \tag{3.7}$$

A basic fact is that $L_{pol}SL(2,\mathbb{C})$ is generated by $SL(2,\mathbb{C})$ $i_0(SL(2,\mathbb{C}))$ (see (5.2.5) of [PS]; $\mathbf{w}_0$ is perhaps a distraction in this context; it appears here to align our notation with that in §5, and its significance is somewhat illuminated by Appendix B).

(3.8) Lemma. *Suppose that* $h = \begin{pmatrix} d & cz^{-1} \\ bz & a \end{pmatrix} \in i_0(SL(2,\mathbb{C}))$, *and* $g \in \mathbf{L}G$ *is in the top stratum, as in* (3.1)-(3.4).

a) If $\alpha = a + bB_1 \neq 0$, *then* $h \cdot g$ *is also in the top stratum and has factorization*

$$[\begin{pmatrix} d & cz^{-1} \\ bz & a \end{pmatrix} g_- \begin{pmatrix} \alpha & 0 \\ \gamma(z) & \alpha^{-1} \end{pmatrix}] \cdot [\begin{pmatrix} \alpha^{-1} & 0 \\ -\gamma_0 & \alpha \end{pmatrix} g_0] \cdot [g_0^{-1} \begin{pmatrix} 1 & 0 \\ \frac{b}{\alpha} z & 1 \end{pmatrix} g_0 g_+],$$

where $\gamma_0 = \frac{-2abA_1 + b^2(B_2 - A_1B_1)}{\alpha}$ *and* $\gamma(z) = \gamma_0 - bz$.

b) If $\alpha' = d - cC_1 \neq 0$, *then* $g \cdot h^{-1}$ *is in the top stratum and has factorization*

$$[g_- g_0 \begin{pmatrix} 1 & -\frac{c}{\alpha'} z^{-1} \\ 0 & 1 \end{pmatrix} g_0^{-1}] \cdot [g_0 \begin{pmatrix} \alpha'^{-1} & -\beta_0 \\ 0 & \alpha' \end{pmatrix}] \cdot [\begin{pmatrix} \alpha' & \beta(z^{-1}) \\ 0 & \alpha'^{-1} \end{pmatrix} g_+ \begin{pmatrix} d & cz^{-1} \\ bz & a \end{pmatrix}^{-1}]$$

where $\beta_0 = \frac{2cda_1 + c^2(c_2 - a_1c_1)}{\alpha'}$ *and* $\beta(z^{-1}) = \beta_0 + cz^{-1}$.

c) If $a_0 \neq 0$, *then* g^σ *is also in the top stratum, and*

$$(g^\sigma)_- = (g_-)^\sigma \begin{pmatrix} 1 & -B_1 \\ \frac{c_0}{a_0} z^{-1} & 1 - \frac{B_1c_0}{a_0} z^{-1} \end{pmatrix},$$

$$(g^\sigma)_0 = \begin{pmatrix} a_0 + \frac{B_1c_1}{a_0} & \frac{B_1}{a_0} \\ \frac{c_1}{a_0} & \frac{1}{a_0} \end{pmatrix}, \quad (g^\sigma)_+ = \begin{pmatrix} 1 & \frac{b_0}{a_0} z \\ -c_1 & 1 - \frac{c_1b_0}{a_0} z \end{pmatrix} (g_+)^\sigma$$

Proof of (3.8). These are straightforward matrix calculations (they also follow from the more abstract calculations in §5 below).//

Part a) implies

$$B((h \cdot g)_-) = (dB + cDz^{-1})/\alpha \tag{3.9}$$

$$D((h \cdot g)_-) = (bBz + aD)/\alpha \tag{3.10}$$

or in terms of components

$$B_n((h \cdot g)_-) = (dB_n + cD_{n-1})/\alpha \tag{3.11}$$

$$D_{n-1}((h \cdot g)_-) = (bB_n + aD_{n-1})/\alpha \tag{3.12}$$

This leads to the following

Corollary (3.13). *The space of variables $\{B_1, D_{n-1}, B_n\}$ is invariant under the action of h. Define $B'_n = B_n/D_{n-1}$. Then*

$$B'_n(h \cdot g) = \frac{c + dB'_n}{a + bB'_n},$$

i.e. B'_n is equivariant with respect to the action of $i_0(SL(2,\mathbb{C}))$ on $\mathbf{L}G$ and the linear fractional action of $SL(2,\mathbb{C})$ on $\hat{\mathbb{C}}$.

Proof of Theorem (0.1) in the case $K = SU(2)$. Let ν_β denote Wiener probability measure for $L_{C^0}K$ with inverse temperature β (see Appendix A). Via the inclusion $L_{C^0}K \to \mathbf{L}G$, we can view the ν_β as probability measures on the formal completion, and these measures are quasi-invariant with respect to the left and right action of $L_{pol}K$, and the action of σ. This family of measures is asymptotically invariant with respect to these actions, meaning for example that

$$\int |1 - \frac{d\nu_\beta(g^\sigma)}{d\nu_\beta(g)}| d\nu_\beta(g) \to 0 \tag{3.14}$$

as $\beta \to 0$, with similar results for g^σ replaced by $l \cdot g$ or $g \cdot r$, for $l, r \in L_{pol}K$ (see Appendix A).

Lemma (3.15). *The top stratum of $L_{pol}G$ has full measure with respect to ν_β.*

This is true for any $L_{pol}K$-biinvariant measure; see (2.1) of Part I of [Pi1]. We will reproduce this argument below when we consider symmetric spaces (see Lemma (6.5)).

Lemma (3.16). *The measures ν_β have weak limits, as $\beta \to 0$, with respect to $BC = BC(\tilde{\Sigma}_1^{formal})$, bounded continuous functions in the linear Riemann-Hilbert coordinate space*

$$H^1_{formal}(\Delta^*, \mathfrak{g}) \times G \times H^1_{formal}(\Delta, \mathfrak{g}).$$

Proof of (3.16). We will write

$$\theta_- = (\theta_1 + \theta_2 w + ..)dw, \tag{3.17}$$

where $w = -1/z$, and the coordinates $\theta_i \in \mathfrak{g}$. The formula $\theta_- = (\partial g_-)g_-^{-1}$ is equivalent to relations of the form

$$g_1 = \theta_1 \tag{3.18}$$

$$2g_2 = \theta_2 + \theta_1 g_1 = \theta_2 + \theta_1^2 = \theta_2 + det(\theta_1) \tag{3.19}$$

$$3g_3 = \theta_3 + \theta_2 g_1 + \theta_1 g_2 = \theta_3 + \theta_2\theta_1 + \frac{1}{2}\theta_1(\theta_2 + det\theta_1), \tag{3.20}$$

i.e. $ng_n = \theta_n$ plus a polynomial in lower order terms.

Slightly abusing standard terminology, we will say that a coordinate, or more generally a function of such coordinates, say λ, is tight if the mass of the measures $\lambda_*\nu_\beta$ does not escape to infinity, i.e. given $\epsilon > 0$, there is a compact set K_ϵ in the target space of λ such that $\lambda_*\nu_\beta(K_\epsilon) > 1 - \epsilon$ for all β. Sums and products of such tight variables are also tight. By a theorem of Prohorov (see chapter 2, §6, of [Bi]), to show that ν_β has weak limits with respect to $BC(\tilde{\Sigma}_1^{formal})$, it is necessary and sufficient to show that g_0, each of the individual coordinates θ_n, and the corresponding coordinates for θ_+, are tight.

Since the functions B'_n are equivariant with respect to the action of $i_0(SU(2))$, it follows from asymptotic invariance that

$$\lim_{\beta\to 0}(B'_n)_*\nu_\beta = \frac{1}{\mathcal{Z}}(1 + |B'_n|^2)^{-2}dm(B'_n), \tag{3.21}$$

the unique invariant probability for unitary linear fractional transformations. In particular each of the B'_n is tight.

To show that each θ_n is tight, we argue as follows (this is the same argument as in [Pi1], but it is more explicit). We know that B_1 is tight (since we know the distribution of the limit, by (3.21)). The invariance with respect to $SU(2)$ implies that $g_1 = \theta_1$ is tight (From an abstract perspective, B_1 is a linear function on the $\{\theta_1\}$; $SU(2)$ acts irreducibly by the adjoint action on $\{\theta_1\}$, hence the $SU(2)$ orbit of B_1, which consists of linear functions all having the same distributional properties, spans the dual of $\{\theta_1\}$; this implies that θ_1 is tight). This implies that D_1 is tight. Since B'_2 is tight, this implies that $B_2 = D_1 B'_2$ is tight. Since $\theta_2 = 2g_2$ plus a polynomial function of g_1, it follows that β_2 is tight (in fact a special feature

of the $SU(2)$ case is that $\beta_2 = 2B_2$, i.e. there are no lower order terms). The invariance with respect to $SU(2)$ implies that θ_2 is tight. This implies that D_2 is tight; together with the tightness of B_3' this implies B_3 is tight, etc.

The tightness of the coefficients of θ_+ is proved in a similar way.

The new aspect of our argument is a simple way to see that g_0 is tight. Because the $(g_0)_*\nu_\beta$ are $SU(2)$-biinvariant, it suffices to show that a_0 is tight.

The basic observation is that the formula for $(g^\sigma)_0$ implies that

$$\frac{b_0}{a_0}(g^\sigma) = \frac{(B_1/a_0)}{a_0 + B_1 c_1/a_0} = \frac{B_1}{a_0^2 + B_1 c_1} \tag{3.22}$$

is tight. But we already know that B_1 and c_1 are tight, and the limiting distribution of B_1 is in the Lebesgue class. Hence it follows that a_0 is tight. This proves Lemma (3.16). $\square$

We now complete the proof of (0.1) in the following way (this part of the argument will not be repeated in subsequent sections). We identify BC with a space of Borel functions on $\mathbf{L}G$, by extending each such function to be zero on the complement of $\tilde{\Sigma}_1^{formal}$. The Lemma implies that the family $\{\nu_\beta\}$ has weak limits with respect to BC. If ν is a limit point, i.e. for some sequence $\beta_j \to 0$, $\nu_{\beta_j}(f) \to \nu(f)$, for all $f \in BC$, then for $g \in L_{pol}K$, asymptotic invariance implies that $g_*\nu_{\beta_j}(f) \to \nu(f)$ as well. In other words ν_{β_j} has a weak limit with respect to g^*BC, for each $g \in L_{pol}K$. Let $\mathcal{F}$ denote the $L_{pol}K$-invariant space generated by BC, which we can identify with a space of Borel functions on $\mathbf{L}G$. We then have a surjective map $\oplus g^*BC \to \mathcal{F}$ and an injective map

$$\mathcal{F}^* \to \prod_{L_{pol}K} \{g^*BC(\tilde{\Sigma}_1^{formal})\}^*. \tag{3.23}$$

The image of the family $\{\nu_\beta\}$ is precompact in each factor of (3.23), hence the image of the family under the map (3.23) is precompact. Suppose that ν is a limit point. To check that $\nu \in \mathcal{F}^*$, suppose that $\sum g_i^* f_i = 0$ in $\mathcal{F}$, where $f_i \in BC$ and $g_i \in L_{pol}K$, $i = 1, .., n$. Then

$$\nu(\sum g_i^* f_i) = \sum \lim_{\beta_j \to 0} \nu_{\beta_j}(g_i^* f_i) = \lim_{\beta_j \to 0} \nu_{\beta_j}(0) = 0. \tag{3.24}$$

Thus ν is a linear functional on $\mathcal{F}$, and we can interpret ν as a probability measure, supported on $\tilde{\Sigma}_1^{formal}$ (or any of its $L_{pol}K$-translates.

Asymptotic invariance implies that any such limit point will be $L_{pol}K$-invariant. $\square$

Remarks (3.25). (a) There is another approach to the tightness of the variables θ_n, for $n > 1$, which uses the action of σ, rather than the auxiliary variables B'_n. As before it is first necessary to observe that B_1 and c_1 are tight. We use the σ action to get tightness of a_0, hence of g_0. We then move from degree to degree using the action of σ. The basic fact is the following:

$$B(g^\sigma) = A((g_-)^\sigma)(-B_1) + B((g_-)^\sigma)(1 - \frac{B_1 c_0}{a_0} z^{-1}) \tag{3.26}$$

$$= -A(g_-)B_1 + zB(g_-)(1 - \frac{B_1 c_0}{a_0} z^{-1}). \tag{3.27}$$

This implies

$$B_n(g^\sigma) = -A_n B_1 + B_{n+1} - \frac{B_n B_1 c_0}{a_0}. \tag{3.28}$$

Using asymptotic invariance with respect to σ, this formula shows that B_{n+1} will be tight if we know that the coefficients of g_- are tight up to order n.

This argument works without change in the S^2 case, as we will observe in §4.

(b) There is another approach to the tightness of a_0, also using the existence of σ. The formula for g_0^σ shows that a_0 and $1/a_0$ have the same limiting distributional properties (a symmetry which we explore in Appendix B). Thus if a_0 is not tight, then probabilistic mass escapes to $a_0 = 0$. This leads to a contradiction with Corollary (1.14), modulo details that will be handled in Lemma (5.15).

On Calculating the Invariant Measure.

Suppose that μ is a $L_{pol}K$-biinvariant probability measure on $\mathbf{L}G$. We would like to calculate the θ_- distribution of μ.

The proof of Lemma (3.16) suggests the following approach. We know the B_1 distribution. We use $SU(2)$-invariance to determine the θ_1 distribution. This determines the joint distribution of D_1 and B_1. By (3.13) we know the B'_2 distribution. The invariant action (3.13) of $i_0(SU(2))$ then determines the joint distribution of D_1, B_1 and B_2. In this $SU(2)$ case we have the special fact that $2B_2 = \beta_2$ (see (3.19)). The knowledge of the $2B_2 = \beta_2$ distribution, together with $SU(2)$-invariance, then determines the θ_2 distribution. We should then have enough information to compute the joint distribution of θ_1 and θ_2 (see (3.29) below). At least in principle, this determines the joint distribution of D_2 and B_1. Using the action (3.13) again, we determine the joint distribution of D_2, B_1, and B_3. This determines β_3, and we continue in this way.

This strategy depends upon several conjectural uniqueness statements, and in practice the calculations rapidly become prohibitive. Here I will simply summarize a partial result, and state a conjecture.

Proposition (3.29). *The probability measure*

$$\mathcal{Z}^{-1}(1+|\theta_1|^2+\frac{1}{2}|\theta_2|^2)^{-7}dm(\theta_1,\theta_2) \tag{3.30}$$

has the property that it restricts to $SU(2)$-invariant distributions for θ_1 and θ_2, a $i_0(SU(2))$-invariant distribution on $\{B_1, D_1, B_2\}$, and has B_1 and B_2' distributions given by (3.21)*.*

As stated, this is a straightforward calculation. This uses a special fact about $sl(2,\mathbb{C})$, namely that $2B_2 = \beta_2$ (see (3.19)). This proposition suggests, but does not guarantee, that (3.30) is the $\{\theta_1,\theta_2\}$-distribution of μ, because there is not an accompanying uniqueness statement (the $\{\theta_i, i \leq 3\}$-distribution definitely has a more complicated form).

Our conjecture that μ is unique, hence $\mathcal{D}$-invariant, implies the conjecture that the distributions of the matrix coefficients of θ_- are $PSU(1,1)$-invariant. The current state of knowledge regarding $PSU(1,1)$-invariant probability measures on $H^1(\Delta)$ (or more generally, $H^m(\Delta)$) is at the stage of constructing interesting examples ([Pi3]). If this symmetry has some power (and I emphasize that I have little feeling for whether this is true), then (3.29) suggests the following

Possible Conjecture (3.31). *The β-distribution of μ is the probability measure with $\beta_1,..,\beta_N$ distribution*

$$\mathcal{Z}^{-1}(1+|\beta_1|^2+..+\frac{1}{N}|\beta_N|^2)^{-1-3N}dm(\beta_1,..,\beta_N)$$

for each N.

This measure is a quotient of a Gaussian. This suggests that there might be some kind of "free field" (or Verma module type) construction of the θ_- distribution of μ.

Remark (3.32). The measure μ has a deformation μ_l parameterized by "level" l (extensively discussed, but not completely proven to exist, in Part III of [Pi1]). This parameter fits into the formulae above in a very natural way. For example in the case of (3.31), one simply replaces the exponent $1+3N$ by $1+3N+l$; the corresponding measures are finite (and coherent) provided $l > -1$.

§4. The S^2 Case

In this case (0.10) specializes to

$$\begin{array}{ccccc} & & G = SL(2,\mathbb{C}) & & \\ & \nearrow & & \nwarrow & \\ G_{\mathbb{R}} = SU(1,1) & & & & U = SU(2,\mathbb{C}), \\ & \nwarrow & & \nearrow & \\ & & K = U(1) & & \end{array} \tag{4.1}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\Theta} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}. \tag{4.2}$$

and $(\cdot)^*$ is the usual Hermitian conjugate.

The top stratum of $\mathbf{L}(SL(2,\mathbb{C})/SU(1,1))$ consists of $g \in \mathbf{L}SL(2,\mathbb{C})$ as in (3.1)–(3.4) with

$$g_0 = \begin{pmatrix} a_0 & b_0 \\ -\bar{b}_0 & d_0 \end{pmatrix}, \quad a_0, d_0 \in \mathbb{R} \tag{4.3}$$

and $g_+ = g_-^{*\Theta}$, so that $a_i = \bar{A}_i$, $b_i = -\bar{C}_i$, $c_i = -\bar{B}_i$, $d_i = \bar{D}_i$.

The action of σ, from §3, commutes with the pointwise action of Θ and $(\cdot)^*$. Hence σ will act on the formal completion of $L(G/G_{\mathbb{R}})$.

The group $\mathcal{L}_{pol}SO(3)$ is generated by $SU(2)$ (the constants), and the map $\tilde{\sigma} : t \to \begin{pmatrix} e^{i\pi t} & 0 \\ 0 & e^{-i\pi t} \end{pmatrix}$; note that the image of $\tilde{\sigma}$ in $L_{pol}Aut(S^2)_0 = L_{pol}SO(3)$ is σ. This follows from (5.2.5) of [PS].

In this special case we will now write out the action of $L_{pol}U$ on $\mathbf{L}(G/G_{\mathbb{R}})$ in Riemann-Hilbert coordinates. These calculations follow directly from the group case of the preceding section.

Lemma (4.4). *Suppose that* $h = \begin{pmatrix} d & cz^{-1} \\ bz & a \end{pmatrix} \in i_0(SL(2,\mathbb{C}))$, *and* $g \in \mathbf{L}(G/G_{\mathbb{R}})$ *is in the top stratum, as in* (3.1) – (3.4) *and* (4.2). *If* $|a + bB_1|^2 \neq |b|^2 a_0^2$ *and* $a + bB_1 \neq 0$, *then* $g' = h \cdot g \cdot h^{*\theta}$ *is also in the top stratum and has factorization* $g' = g'_- \cdot g'_0 \cdot g'^{*\theta}_-$, *where*

$$g'_- = \begin{pmatrix} d & cz^{-1} \\ bz & a \end{pmatrix} g_- g_0 \begin{pmatrix} 1 & \frac{-\bar{b}}{\bar{a}+\bar{b}\bar{B}_1} z^{-1} \\ 0 & 1 \end{pmatrix} g_0^{-1} \begin{pmatrix} \alpha & 0 \\ \gamma(z) & \alpha^{-1} \end{pmatrix},$$

$$g'_0 = \begin{pmatrix} \alpha^{-1} & 0 \\ -\gamma_0 & \alpha \end{pmatrix} g_0 \begin{pmatrix} \alpha'^{-1} & \beta_0 \\ & \alpha' \end{pmatrix},$$

$$\alpha = a + b(B_1 - \frac{\bar{b}}{\bar{a}+\bar{b}\bar{B}_1}a_0^2) = \frac{|a+bB_1|^2 - |b|^2a_0^2}{\bar{a}+\bar{b}\bar{B}_1},$$

$\gamma_0 = \frac{-2abA_1'+b^2(B_2'-A_1'B_1')}{\alpha}$, *$\gamma(z) = \gamma_0 - bz$, $\alpha' = \bar{a}+\bar{b}\bar{B}_1$, and $\beta_0 = \frac{2cda_1+c^2(c_2-a_1c_1)}{\alpha'}$.*

The action of σ is the same as in the group case.

We will not use the precise formulas for β_0 and γ_0.

Let $\mathbf{b} = \bar{b}/(\bar{a}+\bar{b}\bar{B}_1)$, $g_-' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$. Lemma (4.4) implies the formulas:

$$B(g_-') = [(dB+cz^{-1}D) - (dA+cz^{-1}C)\mathbf{b}a_0^2z^{-1} + (dB+cz^{-1}D)\mathbf{b}\bar{b}_0a_0z^{-1}]/\alpha \quad (4.5)$$

$$D(g_-') = [(bzB+aD) - (bzA+aC)\{\mathbf{b}a_0^2z^{-1}\} + (bzB+aD)\mathbf{b}\bar{b}_0a_0z^{-1}]/\alpha \quad (4.6)$$

or

$$B_n(g_-') = [(dB_n+cD_{n-1}) - (dA_{n-1}+cC_{n-2})\mathbf{b}a_0^2 + (dB_{n-1}+cD_{n-2})\mathbf{b}\bar{b}_0a_0]/\alpha \quad (4.7)$$

$$D_{n-1}(g_-') = [(bB_n+aD_{n-1}) - (bA_{n-1}+aC_{n-2})\mathbf{b}a_0^2 + (bB_{n-1}+aD_{n-2})\mathbf{b}\bar{b}_0a_0]/\alpha \quad (4.8)$$

Corollary (4.9). *The space of variables $\{a_0, B_1\}$ is invariant under the action of $h \in i_0(SL_2(\mathbb{C}))$; the action is given by*

$$a(g_0') = \frac{a_0}{\alpha\alpha'} = \frac{a_0}{|a+bB_1|^2 - |b|^2a_0^2}, \quad (4.10)$$

$$B_1(g_-') = [dB_1 + c - \frac{d\bar{b}a_0^2}{\bar{a}+\bar{b}\bar{B}_1}]/\alpha. \quad (4.11)$$

The space of variables $\{a_0, B_1, D_1, B_2\}$ is invariant; the action is given by (4.10), (4.11), *and*

$$B_2(g_-') = [dB_2 + cD_1 - (d(-D_1)+c)\mathbf{b}a_0^2 + (dB_1+c)\mathbf{b}\bar{b}_0a_0)]/\alpha \quad (4.12)$$

$$D_1(g_-') = [bB_2 + aD_1 - b(-D_1)\mathbf{b}a_0^2 + (bB_1+a)\mathbf{b}\bar{b}_0a_0]/\alpha \quad (4.13)$$

The abstract explanation for the existence of this equivariant projection, in the case $n = 1$, is given by the diagram (6.10) in §6.

Proof of Theorem (0.13) for $X = S^2$.

Let $\tilde{\nu}_\beta$ denote the projection of the Wiener measure ν_β on $L_{C^0}SU(2)$ to $L_{C^0}S^2$. We view $\tilde{\nu}_\beta$ as a measure on $\mathbf{L}(SL(2,\mathbb{C})/SU(1,1))$.

Lemma (4.14). *$\tilde{\nu}_\beta$ has full measure on the top stratum.*

This is a special case of Lemma (6.5) (I do not know of an argument that takes advantage of the small rank).

As in the group case, it suffices to show that the coordinates g_0, θ_1,.. are tight, relative to the family of measures $\tilde{\nu}_\beta$. We will first show that g_0 is tight.

Recall the Cartan isomorphism

$$U \times_K i\mathfrak{k} \to G/G_{\mathbb{R}} \to \phi(G/G_{\mathbb{R}}) : [\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}] \to$$

$$\begin{pmatrix} a_0 & b_0 \\ -\bar{b}_0 & d_0 \end{pmatrix} = \begin{pmatrix} |a|^2 e^{2x} - |b|^2 e^{-2x} & ab(e^{2x} + e^{-2x}) \\ -\bar{a}\bar{b}(e^{2x} + e^{-2x}) & |a|^2 e^{-2x} - |b|^2 e^{2x} \end{pmatrix}. \tag{4.15}$$

The corresponding U-equivariant projection

$$\phi(G/G_{\mathbb{R}}) \to U/K \tag{4.16}$$

is given by

$$\begin{pmatrix} a_0 & b_0 \\ -\bar{b}_0 & d_0 \end{pmatrix} \to \zeta = \frac{-\bar{b}_0}{a_0 + d_0 + \sqrt{1 + (\frac{a_0 - d_0}{2})^2}} \tag{4.17}$$

$$= \frac{-\bar{b}_0 a_0}{a_0^2 + 1 - |b_0|^2 + \frac{1}{2}(4a_0^2 + (a_0^2 + |b_0|^2 - 1)^2)^{1/2}} \tag{4.18}$$

where $\zeta = -\bar{b}/a$ is the usual affine coordinate for $\mathbb{CP}^1$. To see this, note that (4.15) implies

$$a_0 + d_0 = 2(|a|^2 - |b|^2)ch(2x) = 2(2|a|^2 - 1)ch(2x) \tag{4.19}$$

$$a_0 - d_0 = 2sh(2x), \quad b_0 = 2abch(2x) \tag{4.20}$$

We can solve for $sh(2x)$ and $ch(2x)$ using the second equation (note that $a_0 - d_0$ will be our basic U-invariant quantity; we will use this below). We can then solve the first equation for $|a|^2$ and the last equation for ab. We then divide to obtain the formula above.

Since $\tilde{\nu}_\beta$ is $SU(2)$-invariant, the ζ-distribution of ν_β is the usual invariant measure $\mathcal{Z}^{-1}(1 + |\zeta|^2)^{-2} dm(\zeta)$.

To show that g_0 is tight, it suffices to show a_0, as a map into $\mathbb{R}$, is tight (a_0 and d_0 have the same distribution and $a_0 d_0 = 1 - |b_0|^2$). To accomplish this we will use the action of σ. An important technical point is that the map

$$LSU(2) \to \phi(LS^2) : g \to gg^{-\Theta} \tag{4.21}$$

is σ equivariant, since σ is conjugation by a multivalued loop which is fixed by Θ for each t. This justifies our use of asymptotic invariance results for σ and ν_β.

Using the formula for $(g^\sigma)_0$ in part (c) of (3.8) (which relates $1/a_0$ and d_0), and then conjugating by $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, we see that a_0 has the same limiting distribution properties as $1/a_0$. It thus suffices to show that $1/a_0$, as a map into $\mathbb{R}$, is tight.

Lemma (4.22). *We identify $SL(2,\mathbb{C})/SU(1,1)$ with its ϕ-image. Suppose that we are given a family $\{\lambda_\beta : \beta > 0\}$ of probability measures on $SL(2,\mathbb{C})/SU(1,1)$ which is quasi-invariant and asymptotically invariant with respect to $SU(1,1)$. Then $1/a_0$, as a map into $\mathbb{R}$, is tight.*

Note that in our present context, the $\lambda_\beta = (g_0)_*\nu_\beta$ are $SU(2)$-invariant.

Proof of (4.22). Note that by Corollary (1.14), for each β, a_0 is nonzero a.e. $[\lambda_\beta]$.

By way of contradiction, suppose that $1/a_0$ is not tight. Then there exists an $\epsilon > 0$ such that for all k there is β_k $(\to 0)$ such that $\nu_{\beta_k}\{|a_0| < k^{-1}\} < \epsilon$. We would like to assert that this means that positive probability accumulates on $\{\begin{pmatrix} 0 & b_0 \\ -\bar{b}_0 & d_0 \end{pmatrix} : |b_0|^2 = 1\}$, in the limit $\beta_k \to 0$, and this positive mass will be $SU(2)$-invariant, contradicting Corollary (1.14). However $|a_0| < M$ is not compact, so mass could possibly escape along the noncompact directions of the lower strata. We will use ζ to fill this gap.

We estimate ζ using (4.18):

$$|\zeta| \leq |a_0| \frac{|b_0|}{||b_0|^2 - 1|} = |a_0| \frac{1}{||b_0| - |b_0|^{-1}|}. \tag{4.23}$$

Since ζ is an affine coordinate for the 2-sphere, and represents a $SU(2)$-equivariant map, the ζ-distribution of ν_β, in the limit $\beta \to 0$, is the invariant measure $\mathcal{Z}^{-1}(1 + |\zeta|^2)^{-2} dm(\zeta)$. Thus we can find $M = M_\epsilon$ such that $\nu_\beta\{\frac{1}{M} \leq |\zeta|\} \geq 1 - \epsilon/2$, for all sufficiently small β. So with ν_{β_k} probability $> \epsilon/2$,

$$||b_0| - \frac{1}{|b_0|}| < \frac{M}{k} \quad \Longrightarrow \quad ||b_0| - 1| < \frac{M}{k}. \tag{4.24}$$

In the same way, using the (4.17), we obtain

$$\frac{1}{M} \leq |\zeta| \leq \frac{|b_0|}{|a_0 + d_0|} \quad \Longrightarrow \quad |a_0 + d_0| \leq M|b_0|. \tag{4.25}$$

Thus we do obtain a nontrivial $SU(2)$-invariant measure on the lower strata, in the limit $k \to \infty$, contradicting (1.14). $\square$

We now know that g_0 is tight. To show that the θ_j are tight, we can proceed as in the $SU(2)$-case, for which we offered two arguments. The first is to note that because a_0 is tight, taking $h = \begin{pmatrix} 0 & z^{-1} \\ -z & 0 \end{pmatrix}$ in (4.10), we see that B_1 is tight. Using $SU(2)$-invariance we obtain that θ_1 is tight. This implies a_0, B_1, and D_1 are tight. Using (4.13) with h above, we see that B_2 is tight. We now continue, using (4.8) in general.

The alternative argument is to observe that the σ-action implies that B_1 has the same distributional properties as b_0/a_0, hence that B_1 is tight. The argument then proceeds exactly as in our second argument in the $SU(2)$ case, part (a) of Remark (3.25), to show that θ_- is tight. $\square$

On Calculating the Invariant Measure.

Suppose that ν is a $L_{pol}SU(2)$-invariant probability measure on $\mathbf{L}(SL(2,\mathbb{C})/SU(1,1))$. The distributions which we can a priori compute are ζ in (4.17), and translates of this (which are given by more complicated formulas). This contrasts sharply with the group case, where we a priori know the distribution of relatively simple variables such as the B_n'.

In §7 we will discuss a conjectural formula for the g_0 distribution. In terms of the Cartan and matrix coordinates in (4.15) (with $z = b_0/a_0$), this conjecture reads

$$(g_0)_*\nu = \frac{1}{\mathcal{Z}} sech^3(2x)ch^2(2x)dk \times dx \tag{4.26}$$

$$= \frac{1}{\mathcal{Z}}(1+(\frac{a_0-d_0}{2})^2)^{-3/2}a_0da_0dm(z) \tag{4.27}$$

$$= \frac{1}{\mathcal{Z}}(1+(\frac{(1+|z|^2)a_0^2-1}{2a_0})^2)^{-3/2}d(a_0^2)dm(z) \tag{4.28}$$

[The first expression is from Lemma (7.5); the second is obtained using (4.20) to convert $sech^3(2x)$ into matrix coordinates, and $(C.19)$ of Appendix C to get an expression for the G-invariant measure in terms of a_0 and z].

Now the distribution of $z = b_0/a_0$ is the same as B_1 (using the formula for σ). Therefore the diagonal distribution conjecture implies the conjecture that the B_1 distribution is given by

$$\frac{1}{\mathcal{Z}}\frac{1}{(1+|B_1|^2)^{3/2}}F(\frac{1}{1+|B_1|^2})dm(B_1), \quad F(\rho) = \int_0^\infty \frac{\rho}{(\rho+\frac{(x-1)^2}{x})^{3/2}}dx \tag{4.29}$$

The algebraic part of (4.29) is reminiscent of (3.31) with the level $l = -1/2$ inserted (see Remark (3.32) for the meaning of level).

The function $F(\rho)$ tends to the finite limit $\int_{-\infty}^{\infty}(1+x^2)^{3/2}dx$ as $\rho \downarrow 0$; it is a decreasing function, and asymptotically behaves like ρ^{-1} as $\rho \to \infty$ [To see this observe that the integral outside the interval $|x-1|<\delta$ vanishes in the limit $\rho \to 0$; for x close to 1 one can get rid of the solitary x in the denominator of the integrand of (4.29)].

At this point I am stuck. I do not see how to use $SU(2)$-invariance to generate an expression for the θ_1 distribution, and in thinking about $PSU(1,1)$-invariant measures, I have never come across something like F.

§5. General Group Case

The point of this section is to indicate how to generalize the argument given in §3, especially to identify the analogue of the variables B_n', and the generalization of the distributional symmetry of a_0 and $1/a_0$.

We assume that $\mathfrak{g}$ is simple. In this case there is a highest root θ. Let h_θ denote the coroot, and choose $e_{\pm\theta}$ in the $\pm\theta$ root spaces such that $e_{-\theta}, h_\theta, e_\theta$ satisfy the canonical $sl(2,\mathbb{C})$ relations. We define $i_0 : sl(2,\mathbb{C}) \to L_{pol}\mathfrak{g}$ by

$$i_0(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) = e_\theta z^{-1}, \quad i_0(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) = -h_\theta, \quad i_0(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = e_{-\theta} z. \tag{5.1}$$

We denote the corresponding group homomorphism by the same symbol, $i_0 : SL(2,\mathbb{C}) \to L_{pol}G$. When $G = SL(2,\mathbb{C})$, this agrees with our notation in §3 and §4. The group $L_{pol}G$ is generated by G and the image of i_0, and a similar statement applies with K in place of G ((5.2.5) of [PS]).

Lemma (5.2). *Suppose that* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ *and* $g = g_- \cdot g_0 \cdot g_+$ *is a point in the top stratum of* $\mathbf{L}G$. *Let* $h = i_0(\begin{pmatrix} a & b \\ c & d \end{pmatrix})$, *write*

$$g_- = exp(x_1 z^{-1} + x_2 z^{-2} + ..), \tag{5.3}$$

and $x_n = x_n' + Z_n e_\theta$ *for the partial root space decomposition of* x_n.

(a) If $a + bZ_1 \neq 0$, *then the point*

$$g' = h \cdot \{g_- \cdot g_0 \cdot g_+\}$$

is also in the top stratum, and has triangular factorization $g' = g_-' \cdot g_0' \cdot g_+'$, *where*

$$g_-' = hg_- exp(-\frac{b}{a+bZ_1} e_{-\theta} z)(a+bZ_1)^{h_\theta} l_0^{-1},$$

$$g_0' = l_0(a+bZ_1)^{-h_\theta}g_0, \quad g_+' = g_0^{-1}exp(\frac{b}{a+bZ_1}e_{-\theta}z)g_0g_+,$$

and

$$l_0 = exp(-ab[e_{-\theta}, a^{-ad(h_\theta)}(x_1')] - b^2Z_2e_{-\theta}) \in N^-.$$

(b) If

$$g_+ = exp(y_1z^{-1} + y_2z^{-2} + ..),$$

$y_n = y_n' + W_ne_{-\theta}$, *and* $a + cW_1 \neq 0$, *then the point*

$$g' = \{g_- \cdot g_0 \cdot g_+\} \cdot h$$

is also in the top stratum, and has triangular factorization $g' = g_-' \cdot g_0' \cdot g_+'$, *where*

$$g_-' = g_-g_0exp(\frac{c}{a+cW_1}e_\theta z^{-1})g_0^{-1},$$

$$g_0' = g_0(a+cW_1)^{-h_\theta}u_0,$$

$$g_+' = u_0^{-1}(a+cW_1)^{h_\theta}exp(\frac{-c}{a+cW_1}e_\theta z^{-1})g_+h,$$

and

$$u_0 = exp(-ac[e_\theta, a^{-ad(h_\theta)}(y_1')] - c^2W_2e_\theta) \in N^+.$$

Proof of (5.2). . This is an easy calculation, once one understands the basic idea. In terms of Lie algbras,

$$\{x \in H^0(D^*, \mathfrak{g}) : x(\infty) \in \mathfrak{b}^-\} \oplus \mathbb{C}e_{-\theta}z \tag{5.4}$$

is a parabolic subalgebra, with a semidirect decomposition

$$i_0(sl(2,\mathbb{C})) \propto \{x \in H^0(D^*, \mathfrak{g}) : x(\infty) \in \mathfrak{b}^-, h_\theta^*(x(\infty)) = 0, Z_1 = 0\}. \tag{5.5}$$

(this is explained in more abstract terms in Part I of [Pi1]). By factoring g_- along this decomposition, we can reduce the calculation to one inside $i_0(Sl(2,\mathbb{C}))$.

We therefore organize g' as

$$g' = (\tilde{g}_-\tilde{g}_-(\infty)^{-1}) \cdot (\tilde{g}_-(\infty)g_0) \cdot (g_0^{-1}exp(-\frac{b}{a+bZ}e_{-\theta}z)g_0g_+), \tag{5.6}$$

where

$$\tilde{g}_- = hg_-exp(-\frac{b}{a+bZ_1}e_{-\theta}z) = \tag{5.7}$$

$$\{exp(-a^{-1}\bar{b}e_\theta z^{-1})a^{-h_\theta}\}\{exp(a^{-1}be_{-\theta}z)g_-exp(-Z_1e_\theta z^{-1})exp(-a^{-1}be_{-\theta}z)\}$$
$$\{exp(a^{-1}be_{-\theta}z)exp(Z_1e_\theta z^{-1})exp(-\frac{b}{a+bZ}e_{-\theta}z)\}. \tag{5.8}$$

We have organized the factors (in braces) to show that $\tilde{g}_- \in G(\mathbb{C}((z^{-1})))$.

We calculate

$$\tilde{g}_-(\infty) = a^{-h_\theta}e^{a^{-1}b[e_{-\theta},x_1']+\frac{1}{2}(a^{-1}b)^2[e_{-\theta},[e_{-\theta},x_2]]}(1+a^{-1}bZ_1)^{-h_\theta}$$
$$= e^{ab[e_{-\theta},a^{-ad(h_\theta)}(x_1')]+b^2Z_2e_{-\theta}}(a+bZ_1)^{-h_\theta}. \tag{5.9}$$

When we plug this into (5.6), we obtain part (a).

Part (b) is proven in the same way. □

We now focus on the left action of h on g_-, and seek the analogue of Lemma (3.13).

Suppose that $\pi = \pi_\lambda$ is an irreducible highest weight representation of G, corresponding to the dominant integral weight λ. We will write π_* for the Lie algebra representation. We can then write

$$\pi(g_-) = 1 + g_1z^{-1} + g_2z^{-2} + ..$$
$$= 1 + \pi_*(x_1)z^{-1} + (\pi_*(x_2) + \frac{1}{2}\pi_*(x_1)^2)z^{-2} + .. \tag{5.10}$$

where the $g_n \in \mathcal{L}(V(\pi))$. We will write $g_0 = 1$.

To generalize the discussion in §3 as directly as possible, we suppose that we can choose λ so that $\lambda(h_\theta) = 1$ (for the classical algebras the λ corresponding to the defining representation has this property). Let $v_{-\lambda}$ denote a lowest weight vector. The assumption $\lambda(h_\theta) = 1$ implies that $v_{-\lambda}$ and $\pi_*(e_\theta)v_{-\lambda}$ span a subrepresentation for $i_0(sl(2,\mathbb{C}))$ ($\simeq sl(2,\mathbb{C})$) isomorphic to the defining representation.

Let D and B, (D_n and B_n, resp.) denote the functions of g_- (g_n, resp.) which pick out the $v_{-\lambda}$ and $\pi_*(e_\theta)v_{-\lambda}$ components of $\pi(g_-)v_{-\lambda}$ ($\pi(g_n)v_{-\lambda}$, resp.), where we split off the other components using the essentially unique K-invariant Hermitian form. Note that D and B are formal power series in z^{-1}, and $D_0 = 1$, since $g_0 = 1$ by convention.

Because $v_{-\lambda}$ is lowest weight, and l_0 and $e_{-\theta}$ are lower triangular, using (a) of Lemma (5.2), we obtain

$$\pi(g_-')(v_{-\lambda}) = \pi(h)\pi(g_-)(a+bZ_1)^{\pi_*(h_\theta)}(v_{-\lambda}) \tag{5.11}$$
$$= (a+bZ_1)^{-1}\pi(h)(\sum_{n=0}^{\infty} g_nv_{-\lambda}z^{-n}). \tag{5.12}$$

This implies the following generalization of (3.13).

Lemma (5.13). *For the left action of h, as in (a) of Lemma* (5.2)*, we have*

$$\begin{pmatrix} B(g'_-) \\ D(g'_-) \end{pmatrix} = (a+bZ_1)^{-1} \begin{pmatrix} d & cz^{-1} \\ bz & a \end{pmatrix} \begin{pmatrix} B(g_-) \\ D(g_-) \end{pmatrix},$$

hence for $n \geq 1$,

$$\begin{pmatrix} B_n(g'_-) \\ D_{n-1}(g'_-) \end{pmatrix} = (a+bZ_1)^{-1} \begin{pmatrix} dB_n(g_-) + cD_{n-1}(g_-) \\ bB_n(g_-) + aD_{n-1}(g_-) \end{pmatrix},$$

and

$$B'_n(g'_-) = \frac{c + dB'_n(g_-)}{a + bB'_n(g_-)},$$

where $B'_1 = B_1$, $B'_n = B_n/D_{n-1}$, $n > 1$.

Proof of Theorem (0.1)*.* We first suppose that $\mathfrak{g} \neq E_8$. Assuming this, we can find a representation π, as above, such that $\lambda(h_\theta) = 1$; for the classical algebras we can choose the defining representation, and for the exceptional algebras, other than E_8, we can choose the smallest nontrivial representation.

We can now proceed as in the $SU(2)$ case. We again introduce the Wiener measures parameterized by inverse temperature β. By asymptotic invariance and the $i_0(SU(2))$-equivariance of B'_n in Lemma (5.13), each of the B'_n has the canonical invariant distribution, $\mathcal{Z}^{-1}(1+|B'_n|^2)^{-2}dm(B'_n)$, in the limit $\beta \to 0$, and hence each is tight. The tightness of B_1, and the irreducible action of K on $\mathfrak{g} \simeq \{\theta_1\}$, implies that $\theta_1 = g_1$ is tight. Thus D_1 is tight. Lemma (5.13) now implies that B_2 is tight. This implies that the analogous coefficient of θ_2 is tight. The action of K now implies that θ_2 is tight, and so on.

To avoid excluding E_8, we consider the adjoint representation in place of π above. We then consider the three-dimensional $i_0(sl(2,\mathbb{C})) \subset \mathfrak{g}$ in place of $span\{v_{-\lambda}, \pi_*(e_\theta)v_{-\lambda}\}$, and an affine coordinate analogous to B'_n for the corresponding 2-sphere of (projective images of) lowest root vectors for $sl(2,\mathbb{C})$ inside $\mathbb{P}(i_0(sl(2,\mathbb{C})))$.

One approach to the tightness of g_0 is given in §4.3 of Part III of [Pi1]. Here we will consider an argument akin to that in §3. This will apply provided $\mathfrak{g} \neq G_2, F_4, E_8$. At the end we will give another argument that works in general.

To prove that g_0 is tight, it suffices to show that the projection of g_0 into G modulo a finite subgroup is tight. Thus it suffices to show that $\pi(g_0)$ is tight, for some irreducible (not necessarily faithful) representation $(\pi, V(\pi))$ of G. To show this, we claim it suffices to prove that a single matrix coefficient $\langle \pi(g_0)v, w \rangle$ is tight,

for some $v, w \in V(\pi)$. To see this, note that $K \times K$ leaves each $\pi(g_0)_* \nu_\beta$ invariant, and acts irreducibly on $\mathcal{L}(V(\pi))$. Thus if one matrix coefficient is tight, then all matrix coefficients are tight, because the span of the orbit of this single coefficient will be the entire linear dual of $\mathcal{L}(V(\pi))$. This implies that $\pi(g_0)$ will be tight, viewed as a map into $\mathcal{L}(V(\pi))$ [Thus in the limit $\beta \to 0$, we will definitely obtain $K \times K$-invariant probability measures on this space of matrices, although we do not a priori know they are supported on $\pi(G)$]. Since $\pi(G)$ is an algebraic group, it is cut out by polynomial equations in the matrix coefficients, and hence these polynomials represent tight variables. Since these equations are satisfied for each β, they will be satisfied in the limit. Thus $\pi(g_0)$ will be tight, viewed as a map into $\pi(G)$, and hence g_0 will be tight.

With probability one, for each β, we can uniquely factor g_0, $g_0 = l_0 m_0 a_0 u_0$, where $l_0 \in N^-$, $u_0 \in N^+$, $m_0 \in T$, and $a_0 \in exp(\mathfrak{h}_\mathbb{R})$ (by Corollary (1.14)). The factors l_0 and u_0 are affine coordinates for the flag spaces $K/T \simeq G/B^+$ and $T \backslash K \simeq B^- \backslash G$, respectively. Since each Wiener measure is $K \times K$-invariant, l_0 and u_0 have the unique K-invariant distributions for all β, hence are tight. Similarly m_0 has the uniform distribution for all β. The key issue concerns the noncompact directions, a_0.

In the $SU(2)$ and S^2 cases we proved the tightness of a_0 by observing that $1/a_0$ has the same limiting distributional properties as a_0. A generalization of this is the following

Lemma (5.14). *Write $a_0 = \prod_1^r a_j^{h_j}$, where the h_j are the coroots of the simple positive roots α_j, and r denotes the rank of $\mathfrak{g}$. Suppose that $\mathfrak{g} \neq G_2, F_4, E_8$. Then for some pair i, j, a_i and $1/a_j$ will have the same limiting distribution properties with respect to the ν_β.*

These symmetries arise from outer automorphisms (parameterized by $C(K)$, which is vacuous in the excluded cases). This is discussed in Appendix B.

Now suppose that g_0 is not tight. Then for each j, all the matrix coefficients for π_{Λ_j} will not be tight, and in particular each a_j will not be tight. The Lemma implies that for some j, $1/a_j$ is not tight, i.e. there exists $\epsilon > 0$ such that for all k, there is β_k ($\to 0$) such that $\nu_{\beta_k}\{a_j < \frac{1}{k}\} > \epsilon$.

We claim that this implies that positive probability accumulates on the lower strata of G, as $\beta \to 0$. The argument is essentially the same as in §3. We can reduce to that situation using the following procedure (which will be used again in the proof of Lemma (6.5) below). Introduce the parabolic subgroup $P_{(j)}$ of G which corresponds to the root α_j. Let $P_{(j)}^+$ denote the nilradical, and let $P_{(j)}^-$ denote

the opposite nilpotent subalgebra. We then consider the $i_{\alpha_j}(SL(2,\mathbb{C}))$-equivariant projection $G \to P^-_{(j)}\backslash G/P^+_{(j)}$. This projection is injective on $i_{\alpha_j}(SL(2,\mathbb{C}))$ itself. When we project the ν_β to this space, we obtain $SU(2)\times SU(2)$-invariant measures on $SL(2,\mathbb{C})$ (identified with the image of i_{α_j}).

This now leads to a contradiction with the following Lemma, which will complete the proof of Theorem (0.1), at least in the cases $\mathfrak{g} \neq G_2, F_4, E_8$.

Lemma (5.15). *Suppose that $\{\lambda_\beta : \beta > 0\}$ is a family of probability measures on $SL(2,\mathbb{C})$ quasi-invariant and asymptotically invariant with respect to $SU(2)\times SU(2)$. Then $1/a$, as a map into $\mathbb{R}$, is tight.*

Proof of (5.15). By way of contradiction, suppose that $1/a$ is not tight. Then there exists $\epsilon > 0$ such that for all k, there is β_k ($\to 0$) such that $\lambda_{\beta_k}\{a_j < \frac{1}{k}\} > \epsilon$.

When we factor

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}, \tag{5.16}$$

for each β, z and w will have the standard $\mathcal{Z}^{-1}(1+|\zeta|^2)^{-2}dm(\zeta)$ distributions, in the limit $\beta \to 0$. Thus there is an $M = M_\epsilon$ such that $|z|, |w| < M$ with λ_β-probability $> 1-\epsilon/3$ for all sufficiently small β. Since $b = aw$, $c = az$, these will accumulate positive probability as $\beta \to 0$.

To control d we need to write down an analogue of ζ from §4. This is straightforward, but somewhat messy. We write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = k_1 p, \quad k_1 \in S(2,\mathbb{C}), \quad p > 0, \tag{5.17}$$

$$p^2 = \begin{pmatrix} A & B \\ \bar{B} & D \end{pmatrix} = \begin{pmatrix} |a|^2+|c|^2 & \bar{a}b+\bar{c}d \\ a\bar{b}+c\bar{d} & |b|^2+|d|^2 \end{pmatrix}. \tag{5.18}$$

There is a $k_2 \in SU(2,\mathbb{C})$ which will diagonalize p, or equivalently p^2. We seek an affine coordinate ζ for $k_2 U(1) \in SU(2)/U(1)$. This will represent an equivariant map to the 2-sphere for the diagonal copy of $SU(2)$.

The eigenvalues of p^2 are $\lambda = (A+D\pm((A+D)^2-4))^{1/2}/2$ Then $p^2-\lambda$ equals

$$\begin{pmatrix} \frac{A-D}{2} \pm \sqrt{(\frac{A+D}{2})^2-1} & B \\ \bar{B} & \frac{D-A}{2} \pm \sqrt{(\frac{A+D}{2})^2-1} \end{pmatrix}. \tag{5.19}$$

We can take

$$\zeta = \frac{\bar{B}}{D-A+\sqrt{(A+D)^2-4}} = \frac{a\bar{b}+z(1+bc)}{D-A+\sqrt{(A+D)^2-4}}, \tag{5.20}$$

where we used $ad - bc = 1$ to rewrite $\bar{B}$ in terms of z (from (5.16)) and other variables not including d. We then have

$$D - A + \sqrt{(A+D)^2 - 4} = (a\bar{b} + z(1+bc))/\zeta \tag{5.21}$$

Since $D \geq 0$, we obtain

$$\sqrt{(A+D)^2 - 4} \leq (a\bar{b} + z(1+bc))/\zeta, \tag{5.22}$$

implying

$$(A+D)^2 \leq 4 + \{(a\bar{b} + z(1+bc))/\zeta\}^2 \tag{5.23}$$

In this expression, only $D = |b|^2 + |d|^2$ involves d.

Since ζ represents an equivariant map to the 2-sphere for the diagonal $SU(2)$, ζ has the usual invariant distribution in the limit $\beta \to 0$. Hence we can find $M' = M'_\epsilon$ such that such that $|1/\zeta| < M'$ with λ_β-probability $> 1 - \epsilon/3$ for all sufficiently small β. We now see that with positive probability b, c, and d will remain finite with positive probability as $k \to \infty$. This means that we do obtain a $SU(2)$-invariant measure on the lower strata, in the limit $k \to \infty$, and this contradicts (1.14). $\square$

If $\mathfrak{g} = G_2, F_4$ or E_8, then this argument does not work, because we cannot apply (5.14). In general we consider $a_0^{-\theta}$. This is the "$|a|^2$" variable for $i_0(SL(2,\mathbb{C}))$; this is due to the minus sign appearing in front of h_θ in (5.1) (see (6.11) below). If none of the a_j are tight, then $a_0^{-\theta}$ tends to zero with some positive probability uniform in β. The Lemma (5.15) then applies and we obtain a contradiction. $\square$

§6. General Symmetric Space Case.

We continue to assume that $\mathfrak{g}$ is simple (so that we will be considering the type I symmetric space case). The homomorphism i_0 in (5.1) will continue to play a critical role. Because Θ preserves the triangular decomposition of $\mathfrak{g}$, $\Theta(\theta) = \theta$, and $\Theta(\mathfrak{g}_\theta) = \mathfrak{g}_\theta$, hence $\Theta(e_\theta) = -\epsilon e_\theta$, where $\epsilon = \pm 1$ (it appears that both possibilities are unavoidable; for example for $Sp(n)/U(n)$, it appears that we must have $\epsilon = +1$, and for the real Grassmannian of oriented p-planes in $\mathbb{R}^{p+q}$, it appears we must have $\epsilon = -1$). Thus the extension of Θ to loop space will map the image of i_0 into itself, and when $\epsilon = 1$, this action will be isomorphic to (4.2) via i_0, otherwise it is trivial.

Lemma (6.1). *Suppose that* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$, *and let* $h = i_0(\begin{pmatrix} a & b \\ c & d \end{pmatrix})$. *Suppose also that* $g = g_- \cdot g_0 \cdot g_+$ *is a point in the top stratum of* $\mathbf{L}(G/G_{\mathbb{R}})$, *so that*

$$g_0^{*\Theta} = g_0, \quad \text{and} \quad g_+ = g_-^{*\Theta}.$$

We write g_- *as in* (5.3), *and* $x_n = x_n' + Z_n e_\theta$. *Let*

$$Z_1' = Z_1 - \frac{\epsilon \bar{b}}{\bar{a} + \epsilon \bar{b} \bar{Z}_1} \langle Ad(g_0) e_\theta, e_\theta \rangle / \langle e_\theta, e_\theta \rangle.$$

If both $a + \epsilon b Z_1$ *and* $a + bZ_1'$ *are not zero, then the point*

$$g' = h \cdot \{g_- \cdot g_0 \cdot g_+\} \cdot h^{*\Theta}$$

is also in the top stratum, and the triangular factorization $g' = g_-' \cdot g_0' \cdot g_+'$ *is given by*

$$g_-' = h g_- g_0 exp(\frac{-\epsilon \bar{b}}{\bar{a} + \epsilon \bar{b} \bar{Z}_1} e_\theta z^{-1}) g_0^{-1} exp(\frac{-b}{a + bZ_1'} e_{-\theta} z)(a + bZ_1')^{h_\theta} l_0'^{-1},$$

$$g_0' = l_0'(a + bZ_1')^{-h_\theta} g_0 (\bar{a} + \epsilon \bar{b} \bar{Z}_1)^{-h_\theta} u_0,$$

$$u_0 = exp(-ac[e_\theta, a^{-ad(h_\theta)}(y_1')] - c^2 W_2 e_\theta) \in N^+,$$

and l_0' *is lower triangular.*

Remark (6.2). Note that for generic g_0, with triangular decomposition $g_0 = l_0 diag(g_0) u_0$, we can write

$$\langle Ad(g_0) e_\theta, e_\theta \rangle / \langle e_\theta, e_\theta \rangle = diag(g_0)^\theta, \tag{6.3}$$

and

$$a + bZ_1' = \frac{|a + bZ_1|^2 + \epsilon b \bar{b} diag(g_0)^\theta}{\bar{a} + \epsilon \bar{b} \bar{Z}_1}, \tag{6.4}$$

These formulas specialize to those in Lemma (4.4) (with $\epsilon = 1$).

Proof of (6.1). This formula is obtained from Lemma (5.2) by first applying $h^{*\Theta}$ on the right, then h on the left (one obtains formulas with considerably different appearance if one does this in the opposite order, and then it is not so easy to compare with (4.4)). $\square$

Proof of Theorem (0.13). Let ν_β denote Wiener measure for $L_{C^0}U$ with inverse temperature β, and let $\tilde{\nu}_\beta$ denote the projection to $L_{C^0}(U/K)$; we view $\tilde{\nu}_\beta$ as a probability measure on $\mathbf{L}(G/G_{\mathbb{R}})$. Because the map $LU \to L(U/K)$: $g \to gg^{-\Theta}$ is equivariant for the left actions of $L_{pol}U$, the measures $\tilde{\nu}_\beta$ form an asymptotically invariant family for this action.

Lemma (6.5). *The top stratum* $\mathbf{L}(G/G_{\mathbb{R}})\cap\tilde{\Sigma}_1^{formal}$ *has full measure with respect to* $\tilde{\nu}_\beta$*.*

Proof of (6.5). The argument is the same as in Proposition (2.1.1) of Part I of [Pi1], for which it is convenient to use Kac-Moody technology. In the proof, for notational clarity, we will suppose $\Theta(e_\theta)=-e_\theta$; we will note the minor changes necessary in the opposite case at the end of the proof.

There is a transitive action

$$G(\mathbb{C}((z^{-1})))\times\mathbf{L}(G/G_{\mathbb{R}})\to\mathbf{L}(G/G_{\mathbb{R}}):g,h\to g\cdot h\cdot g^{*\Theta}. \tag{6.6}$$

This action is well-defined, because if $g\in G(\mathbb{C}((z^{-1})))$, then $g^{*\Theta}\in G(\mathbb{C}((z)))$, and these groups act from the left and right of $\mathbf{L}G$, respectively. We let $\mathcal{B}^{\pm}$ denote the upper and lower Borel subgroups of $G(\mathbb{C}((z^{-1})))$, respectively.

The decomposition for $\mathbf{L}G$ strictly analogous to (1.7) is given by

$$\mathbf{L}G=\bigsqcup_{W\alpha Hom(S^1,T)}\tilde{\Sigma}^L_w,\quad where\quad \tilde{\Sigma}^L_w=\mathcal{B}^-w\mathcal{B}^+, \tag{6.7}$$

and the superscript L is there to remind us of loop space. This induces a decomposition

$$\mathbf{L}(G/G_{\mathbb{R}})=\bigsqcup S_w,\quad where\quad S_w=\tilde{\Sigma}^L_w\cap\mathbf{L}(G/G_{\mathbb{R}}). \tag{6.8}$$

We are mainly interested in S_1, the "top stratum", which consists of g in the formal completion that can be written uniquely as $g=l\cdot h\cdot l^{*\Theta}$, where $l\in\mathcal{N}^-=G(\mathbb{C}[[z^{-1}]])_1$ (the formal completion of the profinite nilpotent algebra spanned by the negative roots of $L_{pol}\mathfrak{g}$), and $h=h^{*\Theta}\in H$. To say that g belongs to S_1 is thus equivalent to the two conditions $g\in\tilde{\Sigma}_1^{formal}$ and $g_0\in\tilde{\Sigma}_1$.

Let $\mathcal{P}^-_{(0)}$ denote the parabolic subgroup, with Levi decomposition,

$$\mathcal{P}^-_{(0)}=i_0(SL(2,\mathbb{C}))\mathcal{B}^-=i_0(SL(2,\mathbb{C}))\alpha\mathcal{R}^-_{(0)}. \tag{6.9}$$

($\mathcal{R}^-_{(0)}$ is the subgroup of $\mathcal{N}^-$ that corresponds to the span of the negative root spaces, $\mathbb{C}e_\theta z^{-1}$ excluded).

We consider the diagram

$$\begin{array}{ccc} \mathbf{L}(G/G_{\mathbb{R}}) & \xrightarrow{q} & \mathcal{R}^-_{(0)}\backslash\mathbf{L}(G/G_{\mathbb{R}}) \\ \uparrow & & \uparrow \\ S_1\cup w_{(0)}S_1 & \to & i_0(\phi(SL(2,\mathbb{C})/SU(1,1))) \\ \uparrow & & \uparrow \\ S_1 & \to & i_0(\tilde{\Sigma}_1) \end{array} \tag{6.10}$$

where $w_{(0)} = i_0(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})$. The map q is $i_0(SL(2,\mathbb{C}))$ equivariant, where the actions in question are induced by (6.6), because this group normalizes $\mathcal{R}^-_{(0)}$. The up arrows are all injective (the only question here might involve the upper right up arrow; if $g, h \in i_0(SL(2,\mathbb{C}))$, $gg^\tau = 1$, $hh^\tau = 1$, and $g = lhl^{*\Theta}$, for some $l \in \mathcal{R}^-_{(0)}$, then $gl^\tau = lh$, an equality in $\mathcal{P}^-_{(0)}$; when we quotient out by $\mathcal{R}^-_{(0)}$, we obtain $g = h$).

The sets on the left of (6.10) are the full q-inverse images of the sets on the right. For if g is in the formal completion and can be written uniquely as $l \cdot h \cdot l^{*\Theta}$, we can factor $l = rl_0$ along the Levi decomposition, and then $g' = l_0 h l_0^{*\Theta} \in i_0(SL(2,\mathbb{C}))$ and $g'g'^\tau = 1$. Conversely if we have a factorization for g', we clearly obtain an element in S_1.

Now suppose that ν is a $L_{pol}K$-quasi-invariant probability measure on the formal completion. Consider $q_*\nu$ restricted to $i_0(\phi(SL(2,\mathbb{C})/SU(1,1))$. This measure (which may have total mass less than one) is $i_0(SU(2))$-quasi-invariant. By Corollary (1.14), it is supported on the top stratum. Therefore the top stratum and its $w_{(0)}$-translate are equivalent, up to a set of $q_*\nu$-measure zero. Together with the fact that the sets on the left of (6.10) are the full q-inverse images of the sets on the right, this implies that the $w_{(0)}$-translate of S_1 and S_1 are equivalent up to a set of ν-measure zero.

Now recall that S_1 consists of $g \in \tilde{\Sigma}_1^{formal} \cap \mathbf{L}(G/G_{\mathbb{R}})$ such that g_0 is in the top stratum of $G/G_{\mathbb{R}}$. By Corollary (1.14), applied to the U-quasi-invariant measure $g_{0*}\nu$, we conclude that S_1 and the top stratum are the same, up to a set of ν-measure zero.

The top stratum is fixed by W. The group generated by $w_{(0)}$ and W is the affine Weyl group, $W \alpha Hom(S^1, T))$. Therefore the translates of the top stratum for the formal completion by the affine Weyl group are all equivalent, up to ν-measure zero. But the union of these translates is the whole space. Therefore the complement of the top stratum must have ν-measure zero.

In the event that $\Theta(e_\theta) = e_\theta$, we replace $SL(2,\mathbb{C})/SU(1,1)$ by $SL(2,\mathbb{C})/SU(2)$, and the same argument applies. $\square$

We now turn to the proof that g_0 is tight. With ν_β probability one, by (1.14), we can write $g_0 = l_0 \mathbf{w} m a_0 l_0^{*\Theta}$, as in (d) of (1.8). As in the group case it suffices to show that in some representation, there is some matrix coefficient which is tight. If we write $a_0 = \prod a_j^{h_j}$, it suffices to show one of the a_j is tight (as a function into $\mathbb{R}$).

As something of a digression, consider first a case in which Θ is an inner automorphism (this is equivalent to $\mathfrak{a}_0 = 0$). In this case the action of Θ on $C(K)$

and $\mathfrak{h}$ is trivial. In the notation of Appendix B, the corresponding automorphism, σ_Δ, will map $\mathbf{L}(G/G_{\mathbb{R}})$ into itself. Lemma (5.14) applies. Thus if all of the a_j are not tight, then for some i, $1/a_i$ is not tight. This leads to a contradiction, using either (4.22) or (5.15), depending upon whether $\Theta(e_j) = \pm e_j$.

Now suppose that Θ is an outer automorphism. In some cases (e.g. $SU(2n)/SO(2n)$), there still exist nontrivial $\Delta \in C(K)$ fixed by Θ. This is illustrated in Appendix B.

In general we argue as follows. If all of the a_j are not tight, this implies that a_0^θ is not tight. Thus $a_0^{-\theta}$ tends to zero with some positive probability uniform in β. But this is the "$|a|$" variable for $i_0(SL(2,\mathbb{C}))$: given an element in $SL(2,\mathbb{C})$ as in (5.16), when we apply i_0, as defined by (5.1),

$$i_0(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}) = a^{-h_\theta} \qquad \Longrightarrow \qquad |a|^2 = a_0^{-\theta}. \tag{6.11}$$

We now consider the diagram as in (6.10). When we push our measures forward, we obtain an asymptotically invariant family of quasi-invariant measures for $SU(2)$ acting on $SL(2,\mathbb{C})/SU(1,1)$ or $SL(2,\mathbb{C})/SU(2)$ (using the isomorphism i_0). We then apply Lemmas (4.22) and (5.15), respectively. This leads to a contradiction.

Thus we know that g_0 is tight. To complete the proof we need to show that the coefficients of θ_-, or equivalently the coefficients of x (where $g_- = exp(x)$), are tight. The argument is the same as the first argument we gave in the S^2 case.

We consider the action of h in Lemma (6.1). The formula for g_0' in Lemma (6.1) implies

$$\langle Ad(g_0')e_\theta, e_\theta\rangle = (a + bZ_1')^{-2}(\bar{a} + \epsilon\bar{b}Z_1)^{-2}\langle Ad(g_0)e_\theta, e_\theta\rangle \tag{6.12}$$

$$= \frac{1}{(|a+bZ_1|^2 + \epsilon|b|^2\langle Ad(g_0)e_\theta, e_\theta\rangle)^2}\langle Ad(g_0)e_\theta, e_\theta\rangle, \tag{6.13}$$

where the second equality uses (6.4). We have already proven that g_0 is tight, so that $\langle Ad(g_0)e_\theta, e_\theta\rangle$ is tight. Asymptotic invariance implies that g_0' (for a given h) is also tight. If we take h with $a = 0$ and $b = 1$, we then see that Z_1 is tight.

Now Z_1 is the e_θ component of x_1. The constants K act irreducibly by the adjoint representation on $\{x_1\}$. Since the constants leave the measures ν_β invariant, it follows that x_1 is tight.

We then apply the formula for g_-' to compute the adjoint matrix coefficient (where Ad is implicit, and we set $\beta = -\epsilon\bar{b}(\bar{a} + \bar{b}\bar{Z}_1)^{-1}$)

$$\langle Ad(g_-')e_{-\theta}, e_{-\theta}\rangle = \langle hg_-g_0exp(\beta e_{-\theta}z^{-1})g_0^{-1}(a + bZ_1')^{-2}e_{-\theta}, e_{-\theta}\rangle) \tag{6.14}$$

We again take h with $a = 0$, $b = 1$. We move h to the right side of the inner product. In the adjoint representation for i_0 we calculate that $h^{-1}e_{-\theta} = -z^{-2}e_\theta$. Also $ad(e_{-\theta})^3 = 0$. We then see that (6.14) equals

$$(Z_1')^{-2}\langle g_- exp(\beta e_{-\theta}^{g_0} z^{-1})e_{-\theta}, -z^{-2}e_\theta\rangle) \tag{6.15}$$

$$= (Z_1')^{-2}\langle(\sum_{n=0}^{\infty} g_n z^{-n})(z^2 + \beta ad(e_{-\theta}^{g_0})z + \frac{1}{2}\beta^2 ad(e_{-\theta}^{g_0})^2)e_{-\theta}, e_\theta\rangle. \tag{6.16}$$

where we are doing this calculation for $z \in S^1$, so that $z^{-1} = \bar{z}$ (these calculations closely mirror, but do not quite reduce to, the calculations $(4.5)-(4.13)$, because we are considering the adjoint representation, rather than the analogue of the defining representation). We now expand the left hand side of (6.11) and (6.16) in powers of z^{-1}. By isolating the terms of order $-n$, we obtain

$$\langle g_n' e_{-\theta}, e_{-\theta}\rangle = (\langle g_{n+1}e_{-\theta}, e_\theta\rangle + ..)/(Z_1')^2, \tag{6.17}$$

where the trailing terms involve g_m, $m \le n$.

We know that Z_1' and $g_1 = x_1$ are tight. This implies that one matrix coefficient for g_2 is tight, and hence that a matrix coefficient for x_2 is tight. Using the action of the constants K, we conclude x_2 is tight. We then continue in this same way. □

§7. Diagonal Distribution Conjecture.

For $g \in \phi(G/G_{\mathbb{R}}) \cap \tilde{\Sigma}_1$, we can write $g = l\mathbf{w}ma_\phi l^{*\Theta}$, as in (d) of (1.8). Let $\tilde{\Sigma}_{\mathbf{w}}$ denote the set of g in this intersection for fixed $\mathbf{w} \in T_0^{(2)}/T_0^{(2)} \cap exp(i\mathfrak{a}_0)$. The following is proven in [Pi4].

Proposition (7.1). *Suppose that U/K is Hermitian symmetric or a group. Then for $\lambda \in (i\mathfrak{t}_0)^*$,*

$$\int_{\phi(U/K)\cap\tilde{\Sigma}_{\mathbf{w}}} a_\phi(g)^{-i\lambda} = \frac{1}{M}\prod^{\mathbf{w}} \frac{\langle\delta,\alpha\rangle}{\langle\delta - i\lambda,\alpha\rangle}$$

where δ is half the sum of the positive complex roots, and the product is over positive roots which are imaginary and noncompact with respect $Ad(\mathbf{w})\Theta$. Therefore

$$\int_{\phi(U/K)} a_\phi(g)^{-i\lambda} = \frac{1}{\#\{[\mathbf{w}]\}}\sum_{[\mathbf{w}]}\prod^{\mathbf{w}} \frac{\langle\rho,\alpha\rangle}{\langle\delta - i\lambda,\alpha\rangle}$$

where the sum is over equivalence classes in $T_0^{(2)}/T_0^{(2)} \cap exp(i\mathfrak{a}_0)$ *such that* $\mathbf{w} \in \phi(U/K)$.

Example (7.2). We write $g \in \phi(S^2)$ as $g = \begin{pmatrix} a_0 & b_0 \\ -\bar{b}_0 & a_0 \end{pmatrix}$, where $a_0 \in \mathbb{R}$, $a_0^2 + |b_0|^2 = 1$. We have $a_\phi = \begin{pmatrix} |a_0| & 0 \\ 0 & |a_0|^{-1} \end{pmatrix}$, and $\mathbf{w}$ is ± 1, corresponding to the upper and lower hemispheres. Write $\lambda = \lambda\alpha_1$, where $\alpha_1 = \lambda_1 - \lambda_2$ is unique noncompact root. We calculate

$$\int_{\phi(S^2)\cap\tilde{\Sigma}_\pm} a_\phi^{-i\lambda} = \int_0^1 t^{-i2\lambda} dt = \frac{1}{1-i2\lambda} = \frac{\langle \frac{1}{2}\alpha_1, \alpha_1 \rangle}{\langle \frac{1}{2}\alpha_1 - i\lambda\alpha_1, \alpha_1 \rangle} \tag{7.3}$$

(and note $\rho = \alpha_1$). The fact that the integrals corresponding to $\mathbf{w} = \pm 1$ are the same is peculiar to this example.

Now suppose that we consider $L(U/K)$. Let ν denote an $L_{pol}U$-invariant probability measure on the formal completion. We think of this loop space as a symmetric space, with $\mathfrak{k}$ replaced by $L\mathfrak{k}$, $\mathfrak{p}$ by $L\mathfrak{p}$, and so on. We then write down the analogue of the formula in (7.1) (for the group case, see §4.3 of Part III of [Pi1]). Using the standard product formula for the *sine* function, we obtain the following

Conjecture (7.4). *Given* g *in the top stratum of* $\mathbf{L}(G/G_\mathbb{R})$, *we write* $g_0 = l_0 \mathbf{w} m a_\phi l_0^{*\Theta}$ *as in* (d) *of* (1.8). *Then*

$$\int_{\mathbf{L}(G/G_\mathbb{R})} a_\phi^{-i\lambda} d\nu = \frac{1}{\#\{\mathbf{w}\}} \sum_{\mathbf{w}} \prod^{\mathbf{w}} \frac{sin(\frac{\pi}{g}\langle \delta, \alpha \rangle)}{sin(\frac{\pi}{g}\langle \delta - i\lambda, \alpha \rangle)}$$

where here the inner product is normalized so that a long root has length $\sqrt{2}$.

In the S^2 case it is a routine exercise to explicitly compute $(g_0)_*\nu$ itself.

Lemma (7.5). *In the* S^2 *case, in terms of the coordinates* (4.15), *with* $k = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, *the Conjecture* (7.4) *is equivalent to*

$$(g_0)_*\nu = \mathcal{Z}^{-1} sech(2x) dk \times dx$$

Sketch of the calculation. By (4.19) − (4.20), $a_0 = (2|a|^2 - 1)ch(2x) + sh(2x)$, and by $(C.6)$ of Appendix C, the $SL(2,\mathbb{C})$-invariant measure on $SL(2,\mathbb{C})/SU(1,1)$ is given by $d\eta = cosh^2(2x) dk \times dx$. If $\delta(2x)d\eta$ is the g_0 distribution, then we must compute

$$\int |a_0|^{-i\lambda} = \int |(2|a|^2 - 1)ch(2x) + sh(2x)|^{-i\lambda} \delta(2x) ch^2(2x) dk dx$$

$$= \int_{x=-\infty}^{+\infty} \int_{u=-1}^{1} |uch(2x) + sh(2x)|^{-i\lambda} ch^q(2x) dx du \tag{7.6}$$

where $u = 2|a|^2 - 1$, and we used the fact that the a-distribution of Haar measure for $SU(2)$ is Lebesgue measure on the unit disk. We do the u-integral, and we eventually obtain

$$\frac{1}{1-i\lambda} \int_{-\infty}^{\infty} e^{-i2x\lambda} e^{2x} \delta(2x) ch^2(2x) sech(2x) dx. \tag{7.7}$$

According to Conjecture (7.5), this equals $1/sin(\frac{\pi}{2}(1-i\lambda))$. When we apply the inverse Fourier transform, we find $\delta(2x) = sech^3(2x)$.

In general, to obtain a formula for the g_0 distribution, it is necessary to use DeLorme's inversion formula for the "diagonal transform" of harmonic analysis for U-invariant functions on $G/G_{\mathbb{R}}$ (see [D] and references). This will be considered elsewhere (in the group case one can instead use Harish-Chandra's spherical transform, which is more elementary).

In [Pi2], I described a possible application of the invariant measure for LK to the construction of the state space for the two dimensional sigma model with target K. That discussion remains essentially unchanged when K is replaced by U/K. The conjectural formula above for the g_0 (or zero-mode) distribution leads to the conjecture that (in the large radius limit) the radial part of the zero-mode Hamiltonian for the S^2 sigma model is equivalent to

$$-(\frac{d}{dx})^2 + \frac{1}{4} - \frac{3}{4} sech^2(x), \quad x \in \mathbb{R} \tag{3.10}$$

(the zero-mode Hamiltonian is itself a U-invariant Laplace type operator on $G/G_{\mathbb{R}}$).

One theme of this paper is that, in considering natural measures associated to the loop space $L(U/K)$, Riemann-Hilbert factorization is important, and when U/K is positively curved, the zero-mode becomes the noncompact space $G/G_{\mathbb{R}}$. An amusing example is

$$\begin{array}{ccccc} & & Sp(2,\mathbb{C}) & & \\ & \nearrow & & \nwarrow & \\ Sp(2,\mathbb{R}) & & & & Sp(2) \\ & \nwarrow & & \nearrow & \\ & & K = U(2) & & \end{array} \tag{1.19}$$

where the compact six dimensional space $Sp(2)/U(2)$ (isomorphic to the Grassmannian of oriented 2-planes in $\mathbb{R}^5$, via an exceptional low rank isomorphism) is

embedded in the 10 dimensional space $Sp(2,\mathbb{C})/Sp(2,\mathbb{R})$, with 4 noncompact dimensions.

In connection with sigma models, it is natural ask about loops into other ideal spaces, such as T, a flat torus, $\pi\backslash G_{\mathbb{R}}/K$, a compact locally symmetric space with negative curvature, or a Ricci flat space. In the nonpositively curved cases, in the framework of this paper, compactness fails. This implies that the existence, or lack thereof, of compactness for Wiener measures (for ideal spaces) corresponds to asymptotic freedom, or the lack thereof, in renormalization group analysis of the corresponding sigma models. Whether there is a tangible connection is unclear, as is the question of how to extend this to more general spaces (or to the supersymmetric setting).

Appendix A. Asymptotic Invariance of Wiener Measure

In this paper we will adopt the following

Definition $(A.1)$**.** *Given a group A of automorphisms of a measure space $(M,\mathcal{M})$, and a family $\{\lambda_\beta : \beta > 0\}$ of quasi-invariant probability measures on M, this family is asymptotically invariant if*

$$\int |1 - \frac{d\sigma_*\lambda_\beta}{d\lambda_\beta}| d\lambda_\beta \to 0 \quad as \quad \beta \downarrow 0,$$

for each $\sigma \in A$.

If the measures λ_β have a weak limit (relative to bounded continuous functions, for example, in a topological context), then the limit will represent an invariant object. Given an A-equivariant map $\phi : M \to N$, the family $\{\phi_*\lambda_\beta\}$ will be asymptotically invariant.

In this appendix we will discuss a slight extension of the asymptotic invariance results in Part III of [Pi1].

Let $X = K$, a simply connected compact Lie group, which we view as a Riemannian symmetric space (using our normalized $Ad(K)$-invariant inner product). Let ν_β denote the Wiener probability measure on $L_{C^0}X$ with inverse temperature β.

Proposition (A.2). *Suppose that $\sigma \in \mathcal{L}_{W^1}Aut(X)_0$. Then*

$$\int |1 - \frac{d\nu_\beta(g^\sigma)}{d\nu_\beta(g)}| d\nu_\beta(g) \leq 2^{3/2}\frac{p^2_{T/2}(1)}{p_T(1)}\beta^{1/2}\mathcal{E}(\sigma)^{1/2},$$

where $\mathcal{E}$ denotes the kinetic energy of σ, $\frac{1}{2}\int_0^1 |\sigma^{-1}d\sigma|^2 dt$, $T = 1/\beta$, and p is the heat kernel.

Proof of (A.2). This result is already known if σ is either a left or right multiplication. The essential point is to treat left and right multiplication simultaneously. We first consider unconditioned paths.

Let $\nu = \nu_\beta^{1,*}$ denote the Wiener probability measure on $Path_{C^0}^{1,*}K$. Given $l, r \in Path_{W^1}^{1,*}K$, we compute

$$\int |1 - \frac{d\nu(lgr)}{d\nu(g)}| d\nu(g) \le \int |1 - \frac{d\nu(gr)}{d\nu(g)}| d\nu(g) + \int |\frac{d\nu(gr)}{d\nu(g)} - \frac{d\nu(lgr)}{d\nu(g)}| d\nu(g) \quad (A.3)$$

$$= \int |1 - \frac{d\nu(gr)}{d\nu(g)}| d\nu(g) + \int |1 - \frac{d\nu(lgr)}{d\nu(gr)}| \frac{d\nu(gr)}{d\nu(g)} d\nu(g) \quad (A.4)$$

$$= \int |1 - \frac{d\nu(gr)}{d\nu(g)}| d\nu(g) + \int |1 - \frac{d\nu(lg)}{d\nu(g)}| d\nu(g) \quad (A.5)$$

$$\le 2\beta^{1/2}(\mathcal{E}(r)^{1/2} + \mathcal{E}(l)^{1/2}) \le 2^{3/2}\beta^{1/2}\mathcal{E}(l,r)^{1/2} \quad (A.6)$$

where the inequality in the fourth line uses (4.1.11) of [Pi1].

Suppose that $\Delta \in C(K)$, and l and r satisfy $l(t+1) = l(t)\Delta$, $r(t+1) = r(t)\Delta$, for all t. We let $\sigma(g) = lgr^{-1}$, for $g \in Path^{1,*}K$. Given $k \in K$, let $\nu_\beta^{1,k}$ denote the Wiener probability measure on $Path_{C^0}^{1,k}K$. Note that σ acts on this space. We have the disintegration formula

$$\nu_\beta^{1,*} = \int_K \nu_\beta^{1,k} p_T(k) dk. \quad (A.7)$$

The inequality $(A.3) - (A.6)$ implies

$$\int_K \{\int |1 - \frac{d\nu_\beta^{1,k}(g^\sigma)}{d\nu_\beta^{1,k}(g)}| d\nu_\beta^{1,k}\} p_T(k) dk \le 2^{3/2}(\beta\mathcal{E}(\sigma))^{1/2}, \quad (A.8)$$

where a priori we only know that $\sigma_*\nu_\beta^{1,k}$ has the same measure class as $\nu_\beta^{1,k}$ for *a.e.* k. We need to translate this statement about an average over K to a statement about each k, especially $k = 1$.

The basic idea is that we can express $\nu_\beta^{1,1}$ as an integral,

$$\nu_\beta^{1,1} = \int_K (\nu_{2\beta}^{1,k} * \nu_{2\beta}^{1,k^{-1}}) \frac{p_{T/2}^2(k)}{p_T(k)} dk. \quad (A.9)$$

The argument following (4.1.11) of Part III of [Pi1] applies verbatim with $\sigma(g)$ in place of the left multiplication $g_L g$. $\square$

Remark $(A.10)$. It is important to extend the asymptotic invariance result above in a number of directions. Unfortunately the method in [Pi] depends on the fact that the target is a group (the methods in [MM] are more robust). This explains why Wiener measure for the target U/K has not made any appearance in this paper.

Appendix B. Symmetries of the Diagonal Distribution.

The Group Case.

In this appendix it is convenient to use the Kac-Moody extension $\mathbb{C} \to \hat{L}_{pol}\mathfrak{g} \to L_{pol}\mathfrak{g}$. Our notation will be consistent with [KW], which compiles numerical information we will need. We very briefly recall that $\mathfrak{g}$, which we assume is simple, is generated by $r = rank(\mathfrak{g})$ copies of $sl(2,\mathbb{C})$, with standard bases h_j, e_j, f_j, $1 \le j \le r$, satisfying the Chevalley-Serre relations; $\hat{L}_{pol}\mathfrak{g}$ is generated by $\mathfrak{g}$ (or the r copies of $sl(2,\mathbb{C})$) and one additional copy of $sl(2,\mathbb{C})$, with standard basis $h_0 = c - h_\theta$, $e_0 = e_{-\theta} \otimes z$, $f_0 = e_\theta \otimes z^{-1}$. The Dynkin diagram of $\hat{L}_{pol}\mathfrak{g}$, which is the extended diagram for $\mathfrak{g}$, encodes the (generalized Chevalley-Serre) relations among the different copies of $sl(2,\mathbb{C})$.

The automorphisms of the Dynkin diagram of $\hat{L}_{pol}\mathfrak{g}$ (see Table II of [KW]) correspond exactly to $Out(K)\alpha C(K)$. Given an element of the center, we obtain an outer automorphism of $\hat{L}_{pol}\mathfrak{g}$, and also of $L_{pol}\mathfrak{g}$ (and its completions). There is a canonical way to realize this outer automorphism by permuting the $r+1$ copies of $sl(2,\mathbb{C})$ which generate $\hat{L}_{pol}\mathfrak{g}$. This automorphism preserves the triangular decomposition of $\hat{L}_{pol}\mathfrak{g}$, it commutes with the unitary involution, and it also lifts to an automorphism of $\hat{L}_{pol}G$ (and its completions). At the level of loops, this automorphism is realized by a multivalued loop in $\mathcal{L}_{pol}Aut(K)$ (such as conjugation by $\mathbf{w}_0 \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}$, which we used in $\S 2-3$); see (3.4.2) and (3.4.4) of [PS]. Asymptotic invariance of Wiener measure implies that the limiting distribution properties of $g \in \mathbf{L}G$, and in particular a_0, are invariant under this symmetry. The formulas we obtain below for this symmetry imply Lemma (5.14). This requires a case by case analysis.

With ν_β probability one, $g = g_- \cdot g_0 \cdot g_+$ is in the top stratum of $\mathbf{L}G$, and in turn $g_0 = l_0 m_0 a_0 u_0$, where $l_0 \in N^-$, $m_0 \in T$, $a_0 \in exp(\mathfrak{h}_\mathbb{R})$, and $u_0 \in N^+$. We can write $a_0 = \prod_1^r a_j^{h_j}$, where the h_j are the coroots of the simple positive roots α_j, and

r denotes the rank of $\mathfrak{g}$. In turn we can write

$$a_j = |\sigma_j(\hat{g})/\sigma_0(\hat{g})^{\check{a}_j}| = |\langle \pi_{\Lambda_j}(g_0)v_{\Lambda_j}, v_{\Lambda_j}\rangle|, \tag{B.1}$$

where the σ_j are the fundamental matrix coefficients, viewed as functions on $\hat{\mathbf{L}}G$, $\hat{g} \in \hat{\mathbf{L}}G$ projects to g, the integers $\check{a}_j$ are determined by $h_\theta = \sum \check{a}_j h_j$ (see Table I of [KW]), the Λ_j are the fundamental dominant weights for $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ corresponding to the α_j, and the π_{Λ_j} denote the corresponding irreducible representations (for the first equality in $(B.1)$, see §4.3 of Part III of [Pi1]).

The main point is that in all cases with $C(K) \neq 0$, there is a corresponding automorphism which interchanges σ_0 and some other σ_j, $j > 0$ (see Table II of [KW]; note that automorphisms which fix the 0 node correspond to outer automorphisms of K). This is what leads to the inversion formulae in Lemma (5.14).

Suppose that $K = SU(n)$. In this case $C(SU(n)) = \mathbb{Z}_n\Delta$, where $\Delta = exp(2\pi i/n)$. The generator Δ corresponds to the Dynkin diagram symmetry $\alpha_j \to \alpha_{j+1}$, where j is read mod $n-1$. All $\check{a}_j = 1$. We want to write down the multivalued loop which corresponds to the automorphism of $\hat{L}\mathfrak{g}$ which sends h_j, e_j, f_j to h_{j+1}, e_{j+1}, f_{j+1}, where j is read mod $n-1$. In the defining representation of $sl(n, \mathbb{C})$, where $\mathfrak{h}$ is the diagonal subalgebra, this multivalued loop is given by

$$\sigma_\Delta(t) = \lambda \mathbf{w} \prod_{j=1}^{n-1} e^{2\pi i(\frac{j}{n})th_j}$$

$$= \lambda \begin{pmatrix} 0 & 0 & .. & 0 & exp(-2\pi i\frac{n-1}{n}t) \\ exp(2\pi it/n) & 0 & .. & & \\ 0 & exp(2\pi it/n) & 0 & & \\ .. & & & & \\ 0 & 0 & .. & exp(2\pi it/n) & 0 \end{pmatrix}$$

where $\mathbf{w}$ is the matrix representing the cyclic permutation $(12..n)$, and the factor λ guarantees $det = 1$, and is otherwise irrelevant. It is straightforward to check that conjugation by σ_Δ implements the automorphism described above.

The automorphism of the Kac-Moody extension cyclically permutes $\sigma_0, \sigma_1, .., \sigma_{n-1}$. Together with $(B.1)$, this implies that σ_Δ maps (a_j) to $(a_2/a_1, a_3/a_1, ..a_{n-1}/a_1, a_1^{-1})$. Thus in this case a_1^{-1} has the same limiting distribution properties as a_{n-1}.

In the case of C_l, $C(Sp(l)) = \mathbb{Z}_2\Delta$, where $\Delta = -1$ (as a matrix in the defining representation). This transposes σ_0 and σ_l. Hence (a_j) maps to $(a_1/a_l, ..a_{l-1}/a_l, 1/a_l)$. We have

$$\sigma_\Delta(t) = \mathbf{w} \prod_{j=1}^{l} exp(i\pi tjh_j)$$

where (if we realize C_l as in [V], page 300) $\mathbf{w}$ is a particular representative for the Weyl group element that maps $(\lambda_1, .., \lambda_l)$ to $(-\lambda_l, .., -\lambda_1)$ (so that $h_i \leftrightarrow h_{l-i}$, $i < l$, and $h_l \leftrightarrow -h_\theta$). The formula is

$$\mathbf{w} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}$$

where $A = diag(i, -i, i, ..)$, B has 1's on the antidiagonal, and A and B are $l \times l$ matrices.

In the case of B_l, $C(Spin(2l+1)) = \mathbb{Z}_2\Delta$, the kernel of the projection to $SO(2l+1)$. The corresponding automorphism transposes σ_1 and σ_0; hence (a_j) maps to $(1/a_1, a_2/a_1, .., a_l/a_1)$. We must have

$$\sigma_\Delta(t) = \mathbf{w}exp(i\pi t h_l),$$

where $\mathbf{w}$ is a representative for the Weyl group element that changes the sign of λ_1 (this implies $h_1 \leftrightarrow -h_\theta$).

The other cases are similar.

The Type I Case.

Suppose that there exists $\Delta \in C(U)$ which is fixed by Θ. In terms of the classification of type I symmetric spaces, this applies in all cases except the family $SU(n)/SO(n)$, for odd n (because $SO(n)$ does not have a center for odd n), $E_6/Sp(4)$ (the central element of $Sp(4)$, of order 2, is not contained in $C(E_6) \simeq \mathbb{Z}_3$), E_6/F_4, and all cases in which $U = G_2, F_4$ or E_8 (because these do not have central elements).

Given Δ fixed by Θ, the corresponding automorphism of $\hat{L}\mathfrak{g}$ will commute with the unique lift of Θ which fixes c, the central element. We therefore obtain a loop σ which will act on $L(U/K)$ and our distributional completions. We write $g_0 = l_0 \mathbf{w} m a_0 l_0^{*\Theta}$.

In cases in which Θ is an inner automorphism, $\mathfrak{h}_0 = \mathfrak{t}_0 = \mathfrak{t}$, and the formulae for the action the a_j above apply.

Suppose $U/K = SU(2n)/SO(2n)$, where Θ is realized via the canonical (outer) automorphism which interchanges $\alpha_1 \leftrightarrow \alpha_{2n-1}$, $\alpha_2 \leftrightarrow \alpha_{2n-2}$, ... We have $\Delta = -1$. Then a_0 is of the form

$$a_0 = a_1^{h_1} .. a_n^{h_n} a_{n-1}^{h_{n+1}} .. a_1^{h_{2n-1}}$$

and the automorphism corresponding to $\Delta = -1$ (the n^{th} power of the automorphism corresponding to σ in $(B.2)$) maps this to the sequence of $a_j's$

$$(a_2/a_n, .., a_{n-1}/a_n, a_1/a_n, 1/a_n, a_1/a_n, a_{n-1}/a_n, .., a_2/a_n)$$

Thus $1/a_n$ is equivalent to a_{n-1} in distribution in the limit $\beta \to 0$.

The case $U/K = SU(2n)/Sp(n)$ is similar.

Appendix C. Integral Formulas for $G/G_{\mathbb{R}}$.

Consider the Cartan decomposition

$$\psi : U \times_K i\mathfrak{k} \to G/G_{\mathbb{R}} : [g,x] \to ge^x G_{\mathbb{R}} \tag{C.1}$$

(here K acts on U on the right and on $i\mathfrak{k}$ by the adjoint action, and given $g \in U$, $x \in i\mathfrak{k}$, $[g,x]$ denotes the image point in $U \times_K i\mathfrak{k}$). We want to express the G-invariant measure on $G/G_{\mathbb{R}}$ (which is unique up to a scalar) in terms of these coordinates. To do this we recall that there is a natural U-invariant connection on the vector bundle

$$U \times_K i\mathfrak{k} \to U/K. \tag{C.2}$$

The horizontal subspace at the point $[g,x] \in U \times_K i\mathfrak{k}$ is defined to be the image of the map

$$i\mathfrak{p} \to Hor|_{[g,x]} : \zeta \to \frac{d}{dt}|_{t=0}[ge^{t\zeta}, x]; \tag{C.3}$$

this map depends upon the choice of (g,x), but the image is independent of this choice. Given this horizontal distribution, we obtain a Riemannian structure on our vector bundle, using the inner products on $i\mathfrak{p}$ and $i\mathfrak{k}$ (the vertical direction) induced by the Killing form. We let dV denote the corresponding volume element.

Let $dV_{G/G_{\mathbb{R}}}$ denote the G-invariant volume for $G/G_{\mathbb{R}}$ (this exists because the adjoint action of $G_{\mathbb{R}}$ on $i\mathfrak{g}_{\mathbb{R}}$ admits an essentially unique invariant volume form).

Proposition (C.4). *We have*

$$\psi^*(dV_{G/G_{\mathbb{R}}}) = c \prod_{\alpha>0}^{\mathfrak{p}} cosh^2(\alpha(x_0)) \prod_{\alpha'>0}^{\mathfrak{k}} |\frac{sinh\alpha'(x_0)}{\alpha'(x_0)}|^2 dV([g,x])$$

where $x \in i\mathfrak{k}$ is K-conjugate to $x_0 \in i\mathfrak{t}_0$, and the products are over the positive roots for $i\mathfrak{t}_0$ acting on $\mathfrak{p}^{\mathbb{C}}$ and $\mathfrak{k}^{\mathbb{C}}$, respectively.

Examples (C.5). (a) If $X = S^2$, then

$$dV_{SL(2)/SU(1,1)} = cosh^2(2|x|)dV, \tag{C.6}$$

(b) In the group case $X = K$, there is a more direct formulation of this result. In this case there is a commutative diagram

$$\begin{array}{ccc} (K \times K) \times_{\Delta(K)} i\mathfrak{k} & \to & (K^{\mathbb{C}} \times K^{\mathbb{C}})/\{(g, g^{-*}) : g \in K^{\mathbb{C}}\} \\ \downarrow & & \downarrow \\ K \times i\mathfrak{k} & \to & K^{\mathbb{C}} \end{array} \tag{C.7}$$

where the first vertical arrow is given by

$$(K \times K) \times_{\Delta(K)} i\mathfrak{k} \to K \times i\mathfrak{k} : [(g, h), x] \to (gh^{-1}, 2hxh^{-1}) \tag{C.8}$$

the second vertical arrow is given by $[g, h] \to gh^*$, and the horizontal arrows are of the form $(g, x) \to ge^x$. Note the essential appearance of the "2" in $(C.8)$. In terms of the coordinates $g = ke^{x'}$, for $g \in K^{\mathbb{C}}$,

$$dg = c \prod_{\alpha > 0} |\frac{sinh(\alpha(x'_0))}{\alpha(x'_0)}|^2 dk \times dx' \tag{C.9}$$

This is equivalent to $(C.4)$, because up to conjugation $x' = 2x$ and $sinh(2x) = 2cosh(x)sinh(x)$, and the $\mathfrak{p}$ and $\mathfrak{k}$ roots are the same.

Proof of (C.4). Fix $g \in U$ and $x \in i\mathfrak{k}$. Given these choices, using the map ψ in $(C.1)$, we can identify

$$T(U \times_K i\mathfrak{k})|_{[g,x]} = Hor \oplus Vert = i\mathfrak{p} \oplus i\mathfrak{k} = i\mathfrak{g}_{\mathbb{R}} \tag{C.10}$$

We also have an identification

$$i\mathfrak{g}_{\mathbb{R}} \to T(G/G_{\mathbb{R}})|_{G_{\mathbb{R}}}. \tag{C.11}$$

Thus given our choice of representative ge^x for the coset $ge^xG_{\mathbb{R}}$, we obtain an identification

$$T(G/G_{\mathbb{R}})|_{ge^xG_{\mathbb{R}}} = i\mathfrak{p} \oplus i\mathfrak{k} = i\mathfrak{g}_{\mathbb{R}}, \tag{C.12}$$

where $X \in i\mathfrak{g}_{\mathbb{R}}$ corresponds to

$$\frac{d}{dt}|_{t=0} ge^x e^{tX} \cdot G_{\mathbb{R}}. \tag{C.13}$$

The G-invariant volume form at $ge^xG_{\mathbb{R}}$ corresponds to the $G_{\mathbb{R}}$-invariant volume form on $i\mathfrak{g}_{\mathbb{R}}$ via this identification.

With these identifications understood, we claim that

$$d\psi|_{[g,x]} : i\mathfrak{p} \oplus i\mathfrak{k} \to i\mathfrak{p} \oplus i\mathfrak{k} : \zeta, y \to proj_{i\mathfrak{g}_{\mathbb{R}}}(e^{-adx}(\zeta) + \frac{1-e^{-adx}}{adx}(y))). \qquad (C.14)$$

To verify this we calculate

$$\frac{d}{dt}|_{t=0}e^{-x}g^{-1}ge^{t\zeta}e^{x+ty} = \frac{d}{dt}|_{t=0}(e^{tAd(e^{-x})(\zeta)}e^{-x}e^{x+ty}) \qquad (C.15)$$

$$= e^{-adx}(\zeta) + \frac{1-e^{-adx}}{adx}(y). \qquad (C.16)$$

When we project, we obtain the claim.

Now we observe that since $x \in i\mathfrak{k}$, $ad(x)$ maps $i\mathfrak{g}_{\mathbb{R}}$ to $\mathfrak{g}_{\mathbb{R}}$, and $ad(x)^2$ maps $i\mathfrak{g}_{\mathbb{R}}$ into itself. Thus with respect to the decomposition $i\mathfrak{p} \oplus i\mathfrak{k}$,

$$d\psi|_{[g,x]} = \begin{pmatrix} cosh(adx) & 0 \\ 0 & \frac{sinh(adx)}{adx} \end{pmatrix}. \qquad (C.17)$$

We then have

$$det_{\mathbb{R}}(\frac{sinh(adx)}{adx} : i\mathfrak{k} \to i\mathfrak{k}) = det_{\mathbb{C}}(\frac{sinh(adx)}{adx} : \mathfrak{k}^C \to \mathfrak{k}^{\mathbb{C}}) = \prod_{\alpha>0}^{\mathfrak{k}}(\frac{sinh(\alpha(x_0))}{\alpha(x_0)})^2. \qquad (C.18)$$

and similarly for $cosh(adx)$. This proves $(C.4)$. □

We now consider the triangular decomposition.

Proposition (C.19). *We write an element of the top stratum of $\phi(G/G_{\mathbb{R}})$ as in (d) of (1.8), $g = l\mathbf{w}ma_{\phi}l^{*\Theta}$. In these coordinates*

$$dV = \mathbf{a}_{\phi}^{\rho}dm(\mathbf{a}_0)dm(l)dm(m)dm(\mathbf{w}),$$

where ρ denotes the sum of the positive complex roots.

Proof of (C.19). B^- has open orbits in the top stratum. The G-invariant measure in these orbits will be determined by B^--invariance. The measure dV is clearly B^--invariant. □

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